Post-surjectivity and balancedness of cellular automata over groups

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Introduction

- Cellular automata can be defined over arbitrary groups.
- Properties of CA are often linked to those of the underlying groups.
- We explore balancedness, the combinatorial property corresponding to preservation of the product measure on the space of configurations.
- We also consider a "dual" of pre-injectivity, which we call post-surjectivity.
- We discuss some relevant links between these two properties and reversibility.

Configurations and patterns over groups

Let $G = (G, \cdot, 1_G, {}^{-1})$ be a group and S be a finite nonempty set.

- For $E, M \subseteq G$: $EM = \{x \cdot y \mid x \in E, y \in M\}, E^{-1} = \{x^{-1} \mid x \in E\}$.
- A configuration is a function $c : G \to S$. We set $\mathcal{C} = S^G$.
- $c, e \in \mathcal{C}$ are asymptotic if $\#\{g \in G \mid c(g) \neq e(g)\} < \infty$.
- A pattern is a function $p : E \to S$ with $E \subseteq G, 0 < \#E < \infty$.

$V \subseteq G$ generates G if words over $V \cup V^{-1}$ represent all elements of G.

- The length of $g \in G$ is the minimum length $\|g\|$ of such a word. We set $D_n = \{g \in G \mid \|g\| \leq n\}$.
- This also induces a distance on $\mathcal{C}$ by

$$d_V(c, e) = 2^{-N} \text{ where } N = \inf\{\|g\| \mid g \in G, \, c(g) \neq e(g)\}$$

In this talk we will only consider infinite, finitely generated groups.

Cellular automata over groups

A cellular automaton (CA) over a group G is a triple $\mathcal{A} = \langle S, \mathcal{N}, f \rangle$ where:

- S is a finite set of states with two or more elements.
- The neighborhood $\mathcal{N} = \{\nu_1, \ldots, \nu_m\} \subseteq G$ is finite and nonempty.
- $f : S^m \to S$ is the local update rule.

The global transition function $F_{\mathcal{A}} : \mathcal{C} \to \mathcal{C}$ is defined by the formula

$$F_{\mathcal{A}}(c)(g) = f\left(c(g \cdot \nu_1), \ldots, c(g \cdot \nu_m)\right) \quad \forall g \in G$$

A pattern $q : M \to S$ is a preimage of $p : E \to S$ if $E\mathcal{N} \subseteq M$ and

$$f\left(q(x \cdot \nu_1), \ldots, q(x \cdot \nu_m)\right)) = p(x) \quad \forall x \in E$$

- **Fact:** if every pattern has a preimage, so does every configuration.

$\mathcal{A}$ is pre-injective if distinct asymptotic configurations have distinct images.

Two noteworthy group properties

A group G is amenable if there exists a finitely additive probability measure μ, defined on every subset of G, such that $\mu(gA) = \mu(A)$ for every $g \in G$, $A \subseteq G$.

- Abelian groups, such as $\mathbb{Z}^d$, are amenable.
- The free groups on two or more generators are not amenable.
- **Bartholdi, 2010:** amenable groups are precisely those where Moore's Garden of Eden theorem holds, *i.e.*, surjective CA are pre-injective.

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A group G is surjunctive if every injective CA on G is surjective.

- No non-surjunctive groups are currently known!
- **Gottschalk's conjecture:** every group is surjunctive.

Balancedness

A cellular automaton $\mathcal{A} = \langle S, \mathcal{N}, f \rangle$ over a group G is balanced if:

for every pattern $p : E \to S$ and every finite $M \subseteq G$ with $E\mathcal{N} \subseteq M$ there are exactly $|S|^{|M|-|E|}$ preimages of p over M

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- d-dimensional surjective CA are balanced. (Hedlund, 1969; Maruoka and Kimura, 1976)
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- **Bartholdi, 2010:** There exists a surjective, non-balanced CA on G if and only if G is not amenable.
- F and H balanced $\Rightarrow$ $F \circ H$ balanced.
- F and $F \circ H$ balanced $\Rightarrow$ H balanced.
- $F \circ H$ and H balanced and H reversible $\Rightarrow$ F balanced.
- **Conjecture:** F injective $\Rightarrow$ F balanced?

Theorem 1: Reversible CA are balanced

Let F be the global transition function of a reversible CA.

- Take a neighborhood radius r for both F and F^{-1}, *i.e.*, let both $F(c)(x)$ and $F^{-1}(c)(x)$ be determined by the state of c on xD_r.

Let $p_1, p_2 : D_n \to S$ be patterns.
We prove that p_1 has as many preimages on D_{n+r} as p_2 has.

As p_1, p_2, and n are arbitrary, the CA is balanced.

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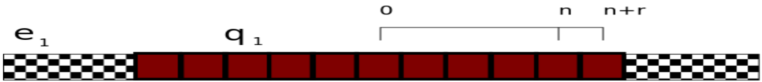
- We define a bijection $T_{1,2}$ between the sets Q_1 and Q_2 of the preimages of p_1 and p_2 on D_{n+r}.
- $T_{1,2}$'s construction is illustrated in the next slide.
  As it only depends on n and r in the general case, the proof for $G = \mathbb{Z}$ works for G arbitrary.
- It turns out that $T_{2,1}$ is the inverse function of $T_{1,2}$!

As p_1, p_2, and n are arbitrary, the CA is balanced.

How $T_{1,2}$ works: a sketch



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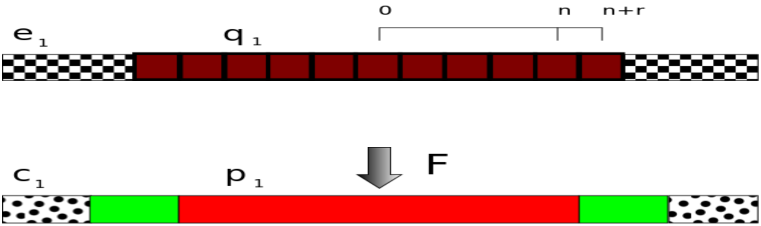

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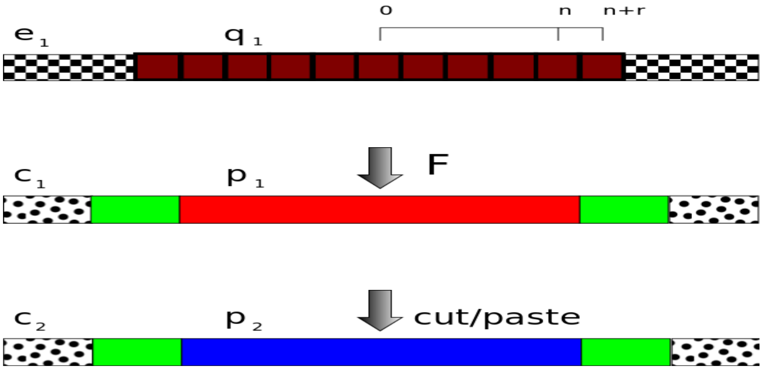

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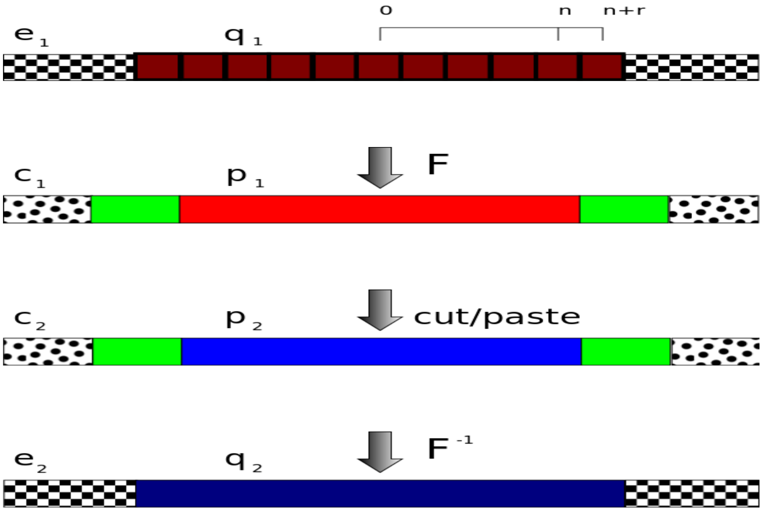

Post-surjectivity

A cellular automaton $\mathcal{A} = \langle S, \mathcal{N}, f \rangle$ over a group G is post-surjective if:

for every $c, e \in \mathcal{C}$ with $F_{\mathcal{A}}(e) = c$
and every $c' \in \mathcal{C}$ asymptotic to c,
there exists $e' \in \mathcal{C}$ asymptotic to e with $F_{\mathcal{A}}(e') = c'$

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Post-surjective CA are surjective.

- Fix $0 \in S$ and take $0' = f(0, \ldots, 0)$.
- A preimage to any pattern p can be found by pasting it on the $0'$-constant configuration.

Not all surjective CA are post-surjective.

- The XOR with the right-hand neighbor is surjective ...
- ... but $\ldots 00100 \ldots$ has no 0-finite preimage.

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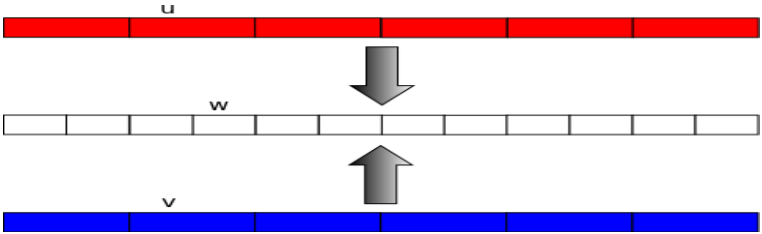
Step 1: characterization of non-reversible 1D CA



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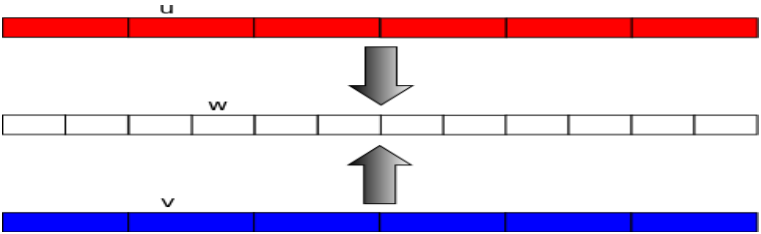
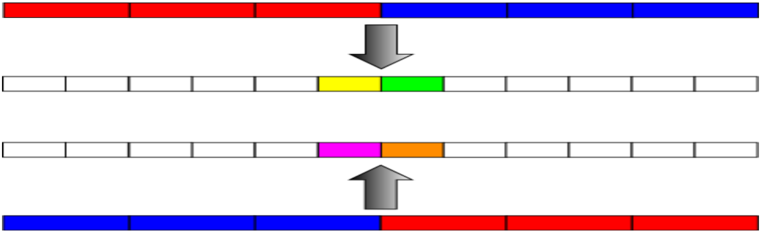
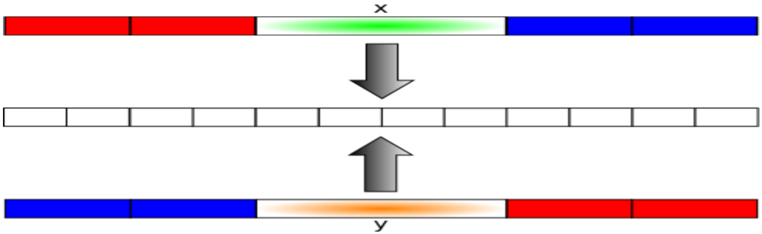


Step 2: swap of halves



Theorem 2: Post-surjective 1D CA are reversible (cont.)

Step 3: post-surjectivity



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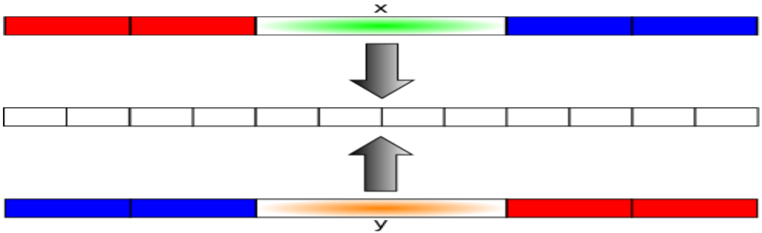
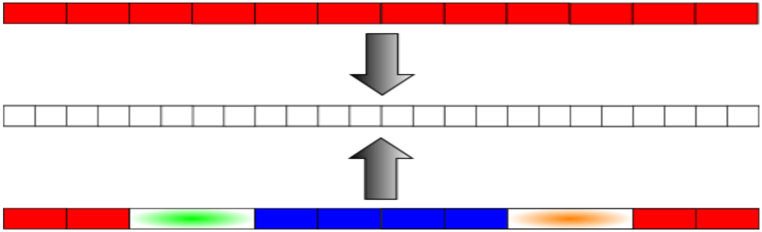


Step 4: contradiction of Moore's Garden of Eden theorem



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To prove Theorem 2, we have made use of two classical results:

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1. a 1D CA being reversible iff it is injective on periodic configurations
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which do not hold in the general case. However:

- pre-injectivity is a bit less than injectivity
- post-surjectivity is a bit more than surjectivity

So may it be that such exchange of power allows to recover reversibility?

Yes, we can! (under mild constraints)

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- Let G be a surjunctive group.
  Let F the global function of a pre-injective, post-surjective CA on G.
- Pre-injectivity and post-surjectivity together yield:
  There exists a finite $M \subseteq G$ such that, for every $c, c' \in \mathcal{C}$,
  if $F(c)$ and $F(c')$ disagree at most on a finite $D \subseteq G$,
  then e and e' disagree at most on DM.

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  if $F(c)$ and $F(c')$ disagree at most on a finite $D \subseteq G$,
  then e and e' disagree at most on DM.
- This allows constructing a CA with neighborhood M^{-1} whose global function H satisfies $F \circ H = \mathrm{id}_{\mathcal{C}}$.
- But a right inverse of a surjective function is injective ...
- ... then H is also surjective by surjunctivity of G, and F is its inverse.

Conclusions and future work

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Open questions:

1. Are injective CA on arbitrary groups balanced?
   This is equivalent to Gottschalk's conjecture.
2. Are there any CA that are post-surjective, but not pre-injective?
   Any such examples must be on a non-amenable group.
3. Are other such transfers possible?

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Thank you for attention!

Any questions?