Recognition of Linear-Slender Context-Free Languages by Real Time One-Way Cellular Automata

Véronique Terrier

GREYC - Université de Caen

Introduction

Objective

To better understand the algorithmic capacity of cellular automata

We will show how one of the simplest types of cellular automata can simulate a restricted variant of context free languages.

Context-free language (CFL)

$\{\mathtt{a}^{\mathtt{n}}\mathtt{b}^{\mathtt{n}+\mathtt{m}}\mathtt{a}^{\mathtt{m}} \colon \mathtt{n} \geq 0, \mathtt{m} \geq 0\}$

$$\begin{aligned} S &\to XY \\ X &\to aXb \mid \varepsilon \\ Y &\to bXa \mid \varepsilon \end{aligned}$$

Linear CFL

$\mathtt{L}_0 = \{\mathtt{a}^{\mathtt{u}}\mathtt{w}\mathtt{b}^{\mathtt{u}} \colon \mathtt{u} > 0, \mathtt{w} = \varepsilon \text{ or } \mathtt{w} \in \mathtt{b}\{\mathtt{a}, \mathtt{b}\}^*\mathtt{a}\}$

$$\begin{aligned} S &\to aSb \mid abTab \mid ab \\ T &\to Ta \mid Tb \mid \varepsilon \end{aligned}$$

Poly-slender and linear-slender CFL

The counting function of a language L

$\sharp_n(L)$: the number of words in L of length n

A language L is

- *k-poly-slender* if $\sharp_n(L)$ is in $\mathcal{O}(n^k)$.
- *linear-slender* if $\sharp_n(L)$ is in $\mathcal{O}(n)$.

Examples

$\{\mathtt{a}^{\mathtt{n}}\mathtt{b}^{\mathtt{n+m}}\mathtt{a}^{\mathtt{m}} : \mathtt{n} \geq 0, \mathtt{m} \geq 0\}$ is a linear-slender CFL

$$\sharp_n(L) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ n/2 - 1 & \text{if } n \text{ is even} \end{cases}$$

$\mathtt{L}_0 = \{\mathtt{a}^{\mathtt{u}}\mathtt{w}\mathtt{b}^{\mathtt{u}} : \mathtt{u} > 0, \mathtt{w} = \varepsilon$ or $\mathtt{w} \in \mathtt{b}\{\mathtt{a}, \mathtt{b}\}^*\mathtt{a}\}$ is not a poly-slender CFL

$\sharp_n(L_0) \sim 2^n$

$\{\mathtt{a}^{\mathtt{n}_1}\mathtt{b}^{\mathtt{n}_2}\mathtt{a}^{\mathtt{n}_3}\mathtt{b}^{\mathtt{n}_4}\mathtt{a}^{\mathtt{n}_5} : \mathtt{n}_1 + \mathtt{n}_3 + \mathtt{n}_5 = \mathtt{n}_2 + \mathtt{n}_4\}$ is a 3-poly-slender CFL

$\sharp_n(L) \sim n^3$

Poly-slender and linear-slender CFL

Characterization in terms of Dyck Loops

k-Dyck Loop (Ilie, Rozenberg and Salomaa)

Given

- a Dyck word on $\{[,]\}$: $z_1 z_2 \cdots z_{2k}$
- some words $y_0, y_1, \cdots, y_{2k}$ and $x_1, \cdots, x_{2k}$
- a map
  $h_{n_1,\cdots,n_k}(y_0 z_1 y_1 z_2 y_2 \cdots z_{2k} y_{2k}) = y_0 x_1^{e_1} y_1 x_2^{e_2} y_2 \cdots x_{2k}^{e_{2k}} y_{2k}$
  where, if z_l and z_r are the i-th matching parenthesis then both exponents e_l and e_r are n_i.

A k-Dyck loop is

$$\{h_{n_1,\cdots,n_k}(y_0 z_1 y_1 z_2 y_2 \cdots y_{2k-1} z_{2k} y_{2k}) \colon n_i \geq 0\}$$

Poly-slender and linear-slender CFL

Characterization in terms of Dyck Loops

Examples

$\{a^{n_1}b^{n_1+n_2}a^{n_2} : n_1, n_2 \geq 0\}$ is a 2-Dyck loop with Dyck word [][], $y_0 = \cdots = y_4 = \varepsilon$ and $x_1 = x_4 = a, x_2 = x_3 = b$

$\{a^{n_1}b^{n_1+n_2}a^{n_2+n_3}b^{n_3+n_4}a^{n_4} : n_i \geq 0\}$ is a 4-Dyck loop with Dyck word [][][][], $y_i = \varepsilon$ and $x_1 = x_4 = x_5 = x_8 = a, x_2 = x_3 = x_6 = x_7 = b$

$\{a^{n_1+n_2}b^{n_2}a^{n_3}b^{n_1+n_3+n_4}a^{n_4} : n_i \geq 0\}$ is a 4-Dyck loop with Dyck word [[][]][]

Theorem (Ilie, Rozenberg and Salomaa)

For any $k \geq 0$, a CFL is k-poly-slender if and only if it is a finite union of $(k+1)$-Dyck loops.

Real time OCA

A real time OCA is specified by:

- an input alphabet Σ

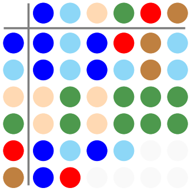
- a finite set of states S $\quad (S \supset \Sigma)$

- a subset of accepting states F

- a transition function $\delta : S \times S \to S$



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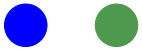

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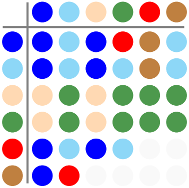

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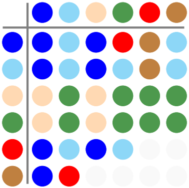

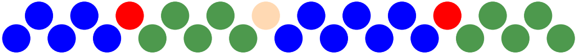

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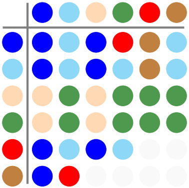

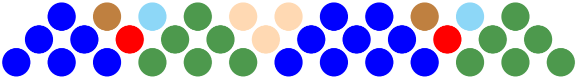

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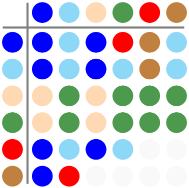
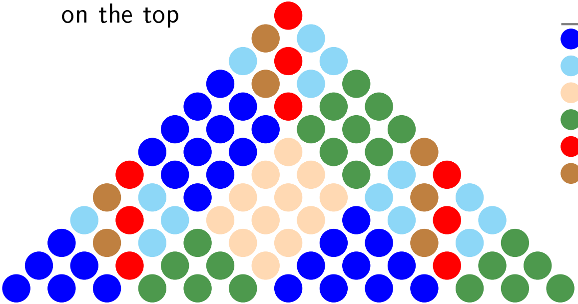


Result of the computation on the top

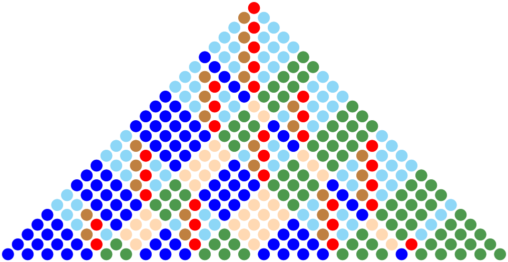


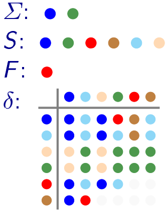
Real time OCA

Language recognition

The language accepted by a real time OCA

The set of all words whose computation ends in an accepting state





accepts

$$L_0 = \{a^u w b^u : u > 0,\ w = \varepsilon \text{ or } w \in b\{a,b\}^*a\}$$

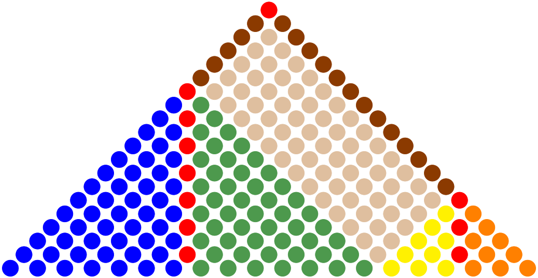
A main feature

The computation of a word contains the computation of all its factors.

Real time OCA

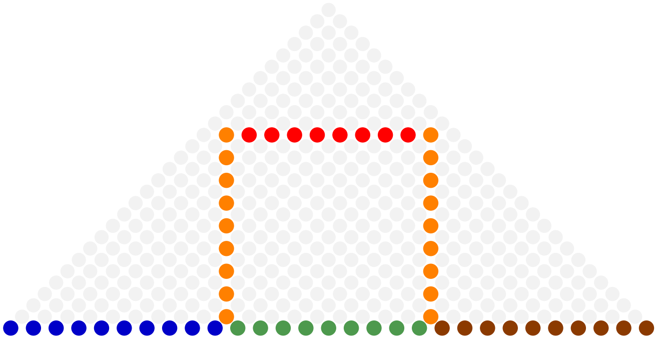
Recognition of

$\{a^n b^n c^m d^m : n \geq 0, m \geq 0\} = \{a^n b^n (c+d)^+ : n \geq 0\} \cap \{(a+b)^+ c^m d^m : m \geq 0\}$



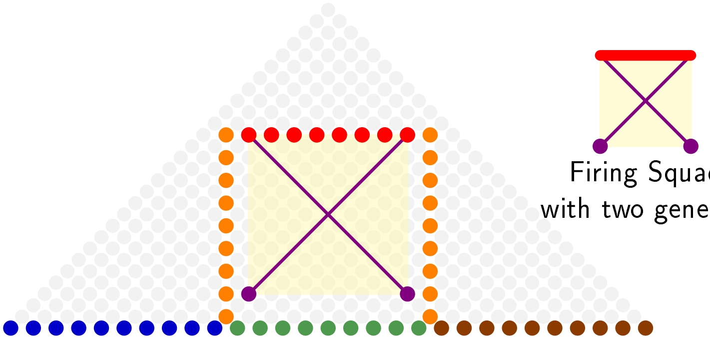
The Čulík algorithm

Recognition of $\{\mathtt{a}^{\mathtt{n}}\mathtt{b}^{\mathtt{n+m}}\mathtt{d}^{\mathtt{m}} : \mathtt{n} \geq 0, \mathtt{m} \geq 0\}$



The Čulík algorithm

Recognition of $\{a^n b^{n+m} d^m : n \geq 0, m \geq 0\}$



Firing Squad
with two generals

Real time OCA

A robust class

The real time OCA class

- is closed under boolean operations
- is not closed under morphism, concatenation
- contains all linear CFL

Characterization in terms of linear conjunctive grammar

A linear CF grammar broaden with a conjunctive operation &

Theorem (Okhotin 2004)

A language L is recognized in real time by an OCA if and only if L is generated by a linear conjunctive grammar.

Relationship between CFL and real time OCA

CFL and real time OCA are incomparable

CFL do not contain real time OCA

$\{a^n b^n c^n : n > 0\}$ is a real time OCA language but not a CFL.

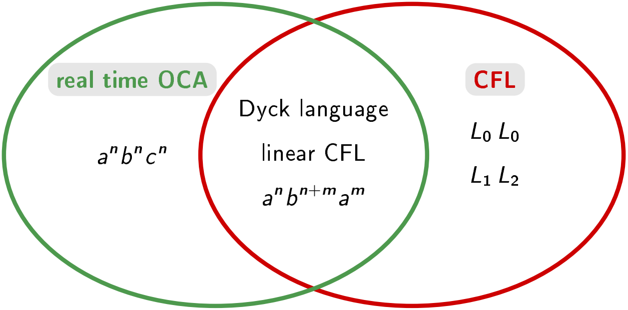
Real time OCA do not contain CFL

$L_0 = \{a^u w b^u : u > 0, w = \varepsilon \text{ or } w \in b\{a,b\}^* a\}$
L_0 is a linear CFL and so a real time OCA language.
$L_0 L_0$ is a CFL but not a real time OCA language.

Real time OCA do not contain deterministic CFL (and even *LL*(1) languages) (Okhotin 2014)

$L_1 = \{c^m b a^{l_1} b \cdots a^{l_{m-1}} b : m > 0, l_i \geq 0\}$
$L_2 = \{a^n b w d^n : n > 0, w \in \{a,b\}^*\}$
L_1 and L_2 are linear CFL and so real time OCA languages.
$L_1 L_2$ is a deterministic CFL (and even a *LL*(1) language) but not a real time OCA language.

Relationship between CFL and real time OCA



Question

What languages do they have in common?

Poly-slender CFL and rt OCA language

Conjecture

The poly-slender CFL are real time OCA languages.

Here we only prove the minimal statement:

Claim

The linear-slender CFL are real time OCA languages.

Recall that the linear-slender CFL correspond to the finite unions of 2-Dyck loops.
$\Rightarrow$ We have to show that the 2-Dyck loops are recognizable by real time OCA.

Poly-slender CFL and rt OCA language

The real question: the closure under concatenation

Theorem (Ginsburg and Spanier)

The family of poly-slender CFL is the smallest family which contains all finite languages and is closed under the following operations:

1. union
2. catenation
3. $(x, y)^\star L = \bigcup_{n \geq 0} x^n L y^n$ for any x, y words

The real time OCA languages

- include all finite languages
- are closed under union and $\star$ operation
- are not closed under concatenation

But, the known witnesses for non-closure under concatenation are not languages inside the family of poly-slender CFL.

2-Dyck loops are real time OCA

Shapes of 2-Dyck loops

Case 1. The underlying Dyck word is [[]]

$$\{y_0 x_1^{n_1} y_1 x_2^{n_2} y_2 x_3^{n_2} y_3 x_4^{n_1} y_4 : n_1, n_2 \geq 0\}$$

Case 2. The underlying Dyck word is [][]

$$\{y_0 x_1^{n_1} y_1 x_2^{n_1} y_2 x_3^{n_2} y_3 x_4^{n_2} y_4 : n_1, n_2 \geq 0\}$$

In the following, we will deal with simplified version of 2-Dyck loops by setting $y_0 = \cdots = y_4 = \varepsilon$.
It does not change the essential arguments.

2-Dyck loops are real time OCA

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2-Dyck loops are real time OCA

Case 1. The underlying Dyck word is [[]]

The corresponding Dyck loop is $\{\mathtt{x}_1^{\mathtt{n}_1}\mathtt{x}_2^{\mathtt{n}_2}\mathtt{x}_3^{\mathtt{n}_2}\mathtt{x}_4^{\mathtt{n}_1} : \mathtt{n}_1, \mathtt{n}_2 \geq 0\}$.
It is a linear CFL language and so it is a real time OCA language.

The closure under $\star$ operation $\left((x, y)^\star L = \bigcup_{n \geq 0} x^n L y^n\right)$ is here involved.

2-Dyck loops are real time OCA

Case 2. The underlying Dyck word is [][]

The corresponding Dyck loop is $\{x_1^{n_1} x_2^{n_1} x_3^{n_2} x_4^{n_2} : n_1, n_2 \geq 0\}$.

The closure under concatenation of 1-Dyck loops is now involved.

Case 2.a. x_1 and x_2 have the same primitive root (or x_3 and x_4)

A degenerated case

$x_1 = z^r, x_2 = z^s$ for some z, r, s

$\{x_1^{n_1} x_2^{n_1} x_3^{n_2} x_4^{n_2} : n_1, n_2 \geq 0\} = \{z^{(r+s)n_1} x_3^{n_2} x_4^{n_2} : n_1, n_2 \geq 0\}$ is a linear CFL

Case 2.b. x_2 and x_3 do not have the same primitive root

A type of marked concatenation

The Dyck loop is the intersection of two linear CFL

$$\{x_1^{n_1} x_2^{n_1} x_3^{n_2} x_4^{n_2} : n_1, n_2 \geq 0\} = \{x_1^{n_1} x_2^{n_1} x_3^{m} x_4^{p} : n_1, m, p \geq 0\} \cap \{x_1^{m} x_2^{p} x_3^{n_2} x_4^{n_2} : n_2, m, p \geq 0\}$$

Case 2.c. x_2 and x_3 have the same primitive root but not x_1, x_2 and x_3, x_4

The critical case

$\{x_1^{n_1} x_2^{n_1} x_3^{n_2} x_4^{n_2} : n_1, n_2 \geq 0\} = \{x_1^{n_1} z^{r\,n_1 + s\,n_2} x_4^{n_2} : n_1, n_2 \geq 0\}$

2-Dyck loops are real time OCA

Dyck loops of shape $\{x_1^{n_1} z^{r\,n_1+s\,n_2} x_4^{n_2} : n_1, n_2 \geq 0\}$ with x_1 and z having distinct primitive roots as well as z and x_4

- The Čulík's OCA recognizes in real time $\{a^{n_1} b^{n_1+n_2} c^{n_2} : n_1, n_2 \geq 0\}$ with a, b, c distinct letters
- By way of geometric transformations of the Čulík's OCA, we construct real time OCAs which recognize the slight variants: $\{a^{n_1} b^{r\,n_1+s\,n_2} c^{n_2} : n_1, n_2 \geq 0\}$
- Making use of Okhotin's equivalence, we translate these OCAs in terms of linear conjunctive grammars
- Let h be the homomorphism $h(a) = x_1$, $h(b) = z$, $h(c) = x_4$. Providing that $h(a)$ and $h(b)$, as well as $h(b)$ and $h(c)$, have distinct primitive roots, we can modify the previous linear conjunctive grammars to generate $h\left(\{a^{n_1} b^{r\,n_1+s\,n_2} c^{n_2} : n_1, n_2 \geq 0\}\right) = \{x_1^{n_1} z^{r\,n_1+s\,n_2} x_4^{n_2} : n_1, n_2 \geq 0\}$

Conclusion

Linear-slender CFL are real time OCA languages

Three ingredients:

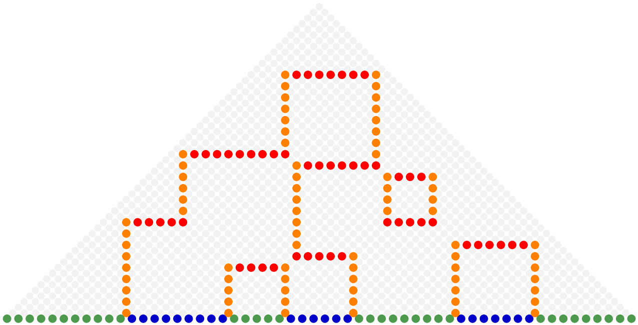
- the characterization of poly-slender CFL in terms of Dyck loops given by Ilie, Rozenberg and Salomaa
- the Čulík's OCA which recognizes in real time the language $\{a^n b^{n+m} a^m : n, m \geq 0\}$
- the Okhotin's characterization of real time OCA by linear conjunctive grammars

Conclusion

Question: Are poly-slender CFL real time OCA languages?

A preliminary step:

> How to extend the Čulík's algorithm for the bounded versions of the language of words whose the number of a's equals the number of b's?



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