On the Periods of Spatially Periodic Preimages in Linear Bipermutive Cellular Automata

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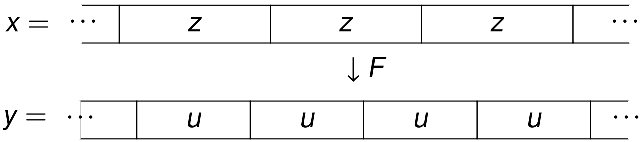
Outline

Spatially Periodic Preimages in Surjective CAs

- Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a (CA) with $|A| = q$, and let $y \in A^{\mathbb{Z}}$ be a spatially periodic configuration of period $p \in \mathbb{N}$ defined by a finite word $u \in A^p$, i.e. $y = {}^{\omega}u^{\omega}$
- If F is surjective, it is known that each preimage x of y under F is spatially periodic as well [Hedlund73, Cattaneo00]



- What are the periods of preimages $x \in F^{-1}(y)$?

Assumptions and Problem Statement

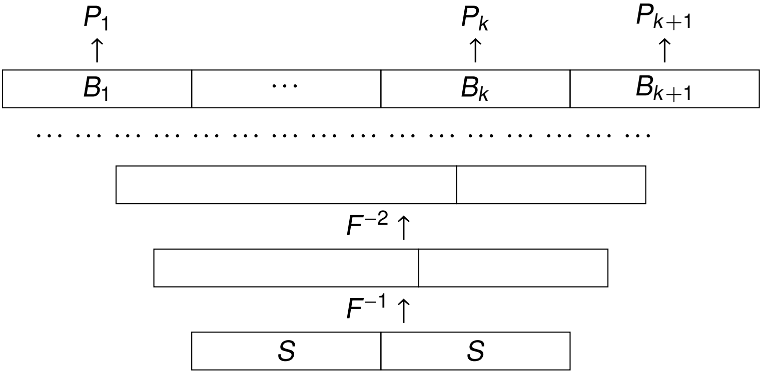
- We focus our attention on the class of bipermutive CA (BCA)
- A CA $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ induced by a local rule $f : A^{2r+1} \to A$ is bipermutive if, by fixing the first (the last) $2r$ coordinates of f, the resulting restriction $f_{R,z} : A \to A$ ($f_{L,z} : A \to A$) is a permutation on A

Problem PBCAP - Periods of BCA Preimages

Let $y \in A^{\mathbb{Z}}$ be a spatially periodic configuration of period $p \in \mathbb{N}$. Given a BCA $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, find the relation between p and the spatial periods of the preimages $x \in F^{-1}(y)$.

Motivation: BCA-based Secret Sharing Scheme

- Motivation for solving PBCAP: find the maximum number of players in a BCA-based Secret Sharing Scheme [Mariot14]



Outline

Problem Statement

Preimages Periods in Generic BCA

Linear BCA Preimages and Concatenated LRS

Conclusions and Future Directions of Research

Preimage Computation in BCA

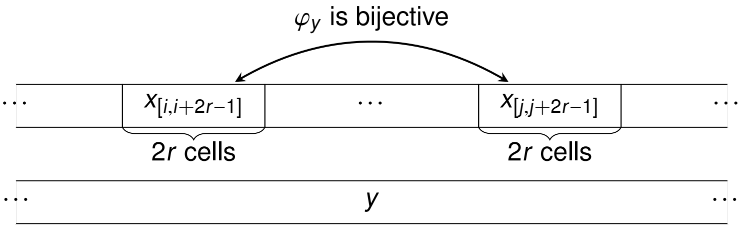
- Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a BCA with local rule $f : A^{2r+1} \to A$, and let $y \in A^{\mathbb{Z}}$ be a configuration
- Additionally, let $x_{[i,i+2r-1]} \in A^{2r}$ be the $2r$-cell block placed at position $i \in \mathbb{Z}$ of a preimage $x \in F^{-1}(y)$
- The remainder of x is determined by the following equation:

$$x_n = \begin{cases} f^{-1}_{R,z(n)}(y_{n-r}), \text{ where } z(n) = x_{[n-2r,n-1]}, & \text{if } n \geq i+2r \quad \text{(a)} \\ \\ f^{-1}_{L,z(n)}(y_{n+r}), \text{ where } z(n) = x_{[n+1,n+2r]}, & \text{if } n < i \quad \text{(b)} \end{cases}$$

Preimages Periods in Generic BCA (1/2)

Lemma

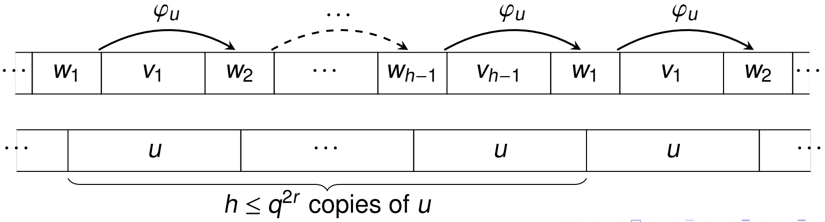
Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a BCA with local rule $f : A^{2r+1} \to A$. Given a configuration $y \in A^{\mathbb{Z}}$ and $i, j \in \mathbb{Z}$, for all $x \in F^{-1}(y)$ there exists a permutation φ_y between the blocks $x_{[i,i+2r-1]}$ and $x_{[j,j+2r-1]}$.



Preimages Periods in Generic BCA (2/2)

Proposition

Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a BCA with local rule $f : A^{2r+1} \to A$ and let $y \in A^{\mathbb{Z}}$ be a spatially periodic configuration of period $p \in \mathbb{N}$. Given a preimage $x \in F^{-1}(y)$, the period of x is $m = p \cdot h$, where $h \in \{1, \cdots, q^{2r}\}$.



Outline

Linear BCA

- We now assume that the alphabet is a finite field, that is, $A = \mathbb{F}_q$ where q is a power of a prime
- A CA $F : \mathbb{F}_q^{\mathbb{Z}} \to \mathbb{F}_q^{\mathbb{Z}}$ is *linear* if its local rule $f : \mathbb{F}_q^{2r+1} \to \mathbb{F}_q$ is a linear combination of the neighborhood $x \in \mathbb{F}_q^{2r+1}$:

$$f(x_0, \cdots, x_{2r}) = c_0 \cdot x_0 + \cdots + c_{2r} \cdot x_{2r} \quad ,$$

  for a certain vector $c = (c_0, c_1, \cdots, c_{2r}) \in \mathbb{F}_q^{2r+1}$
- Remark: if $c_0, c_{2r} \neq 0$, then a linear CA is also bipermutive (LBCA)

Linear Recurring Sequences

- Given $a_0, a_1, \cdots, a_{k-1} \in \mathbb{F}_q$, a linear recurring sequence (LRS) of order k is a sequence $s = s_0, s_1, \cdots$ of elements in $\mathbb{F}_q$ satisfying

$$s_{n+k} = a_0 s_n + a_1 s_{n+1} + \cdots + a_{k-1} s_{n+k-1} \quad \forall n \in \mathbb{N}$$

- A LRS is generated by a Linear Feedback Shift Register (LFSR)
- The characteristic polynomial of s is defined as

$$a(X) = X^k - a_{k-1}X^{k-1} - a_{k-2}X^{k-2} - \cdots - a_0$$

- The period of s equals the order of the minimal polynomial $m(X)$, which depends on $a(X)$ and the initial terms of s

Characterising LBCA Preimages as Concatenated LRS (1/2)

- Given a LBCA F, a preimage $x \in F^{-1}(y)$ of y can be considered as a LRS of order $k = 2r$ "disturbed" by y
- Let $c_0, \cdots, c_{2r}$ be the coefficients of the local rule f, and set
  - $d = c_{2r}^{-1}$
  - $a_i = -d \cdot c_i$ for $i \in \{0, \cdots, 2r-1\}$
- Moreover, define sequence v as the r-shift of y, that is, $v_n = y_{n+r}$ for $n \in \mathbb{N}$
- Case (a) of the preimage recurrence equation becomes

$$x_{n+k} = a_0 x_n + a_1 x_{n+1} + \cdots + a_{k-1} x_{n+k-1} + d v_n \quad \forall n \geq 2r$$

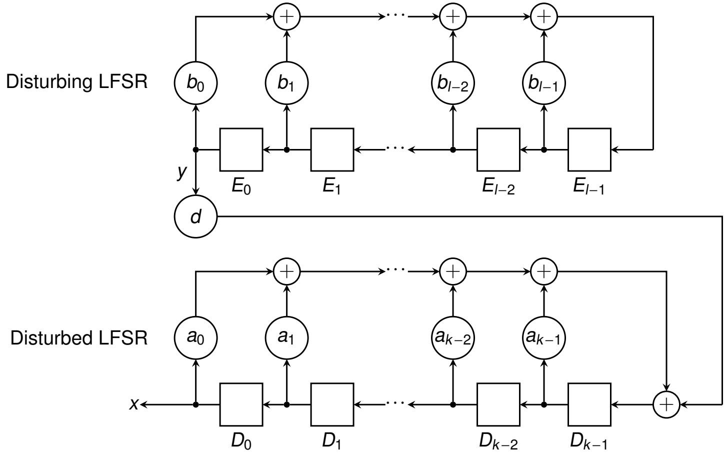
Characterising LBCA Preimages as Concatenated LRS (2/2)

- Remark: If y is spatially periodic of period p, then sequence $v = \{v_n\}_{n\in\mathbb{N}}$ is a LRS of a certain order $l \in \mathbb{N}$:

$$v_{n+l} = b_0 v_n + b_1 v_{n+1} + \cdots + b_{l-1} v_{n+l-1} \quad \forall n \in \mathbb{N}$$

- In the worst case, v will be generated by the "trivial" LRS of order $l = p$ which cyclically shifts a word of length p
- We define x as the concatenation $s \leftrightsquigarrow v$ of the LRS s induced by the local rule f and the LRS v which is the r-shift of y

LBCA Preimage Generation By Concatenated LFSR



Characteristic Polynomial of Concatenated LRS

Theorem

Let $s \leftrightsquigarrow v$ be the concatenation of LRS s and v, and let $a(X), b(X) \in \mathbb{F}_q[X]$ be the characteristic polynomials of s and v. Then, $a(X) \cdot b(X)$ is a characteristic polynomial of $s \leftrightsquigarrow v$.

Proof (Idea):

- Decompose $s \leftrightsquigarrow v$ as the sum of sequence s *without* disturbance and the 0-concatenation $s \leftrightsquigarrow_0 v$, where the LFSR of s is initialised to 0
- Determine the generating function of $s \leftrightsquigarrow_0 v$ [Chassé93], and then apply the fundamental identity of formal power series to find the characteristic polynomial of $s \leftrightsquigarrow v$

Single Preimage Period Computation

Input: An LBCA F with local rule $f : \mathbb{F}_q^{2r+1} \to \mathbb{F}_q$, a spatially periodic configuration $y \in \mathbb{F}_q^{\mathbb{Z}}$ and a block $x_{[0,2r-1]}$ of $x \in F^{-1}(y)$

1. Find the minimal polynomial $b(X) = X^l - b_{l-1}X^{l-1} \cdots - b_0$ of the LRS $v = \{v_n = y_{n+r}\}_{n \in \mathbb{N}}$
2. Set the characteristic polynomial $a(X)$ associated to f to $a(X) = X^k - a_{k-1}X^{k-1} - \cdots - a_0$
3. Compute the characteristic polynomial $c(X) = a(X) \cdot b(X)$
4. Determine the minimal polynomial $m(X)$, using the characteristic polynomial $c(X)$ and the block $x_{[0,2r-1]}$
5. Compute the order of $m(X)$, and output it as the period of x

Periods Characterization for Irreducible Polynomials

Complete characterization of the periods of y when both $a(X)$ and $b(X)$ are irreducible:

Theorem

- *Let $a(X)$ be the characteristic polynomial associated to $f_{R,z}^{-1}$, and suppose that $a(X)$ has order e*
- *Let $y \in \mathbb{F}_q^{\mathbb{Z}}$ be a spatially periodic configuration of period $p > 1$, and let $b(X)$ be the minimal polynomial of $v = \{v_n = y_{n+r}\}_{n\in\mathbb{N}}$*
- *Assume that both $a(X)$ and $b(X)$ are irreducible*

$\Rightarrow F^{-1}(y)$ contains one configuration of period p and $q^k - 1$ configurations of period m, where $m = lcm(e,p)$.

Outline

Results Summary

- When the CA is only bipermutive, the preimages periods of a spatially periodic configuration y are multiple of the period of y
- In the case of LBCA, the preimages periods can be studied in terms of concatenated LRS
- Using the characteristic polynomial of the corresponding concatenated LRS, we derived an algorithm to compute the period of a single preimage
- In the particular case where both the characteristic polynomial induced respectively by the local rule and y are irreducible, we showed a characterization of the periods of all preimages of y

Future Directions

- Generalise the results with respect to the t-th iterate F^t
- Consider nonlinear rules. In this case, the preimage is generated by a Nonlinear Feedback Shift Register (NFSR) disturbed by a LFSR
- Results on the nonlinear case could have an impact on the cryptanalysis of the stream cipher Grain [Hell08]
- Investigate the preimages periods under the action of generic surjective CA and multi-dimensional CA

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