TECHNISCHE UNIVERSITÄT DRESDEN

TOPOLOGICAL CONJUGACY OF ELEMENTARY CELLULAR AUTOMATA

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Definition

Let A be finite with $|A| \geq 2$, then $A^{\mathbb{Z}}$ together with the metric

$$d(x, y) := 2^{-\min\{|k|\,;\, \ell\in\mathbb{Z},\, x_\ell \neq y_\ell\}}$$

is a compact metric space homeomorphic to the Cantor set.

Theorem (Curtis,Hedlund,Lyndon (1969))

A map $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a cellular automaton (it is defined by a block map), if it is continuous and commuting with the left shift σ.

Isomorphisms of topological dynamical systems

- F, H continuous

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "\Phi"'] & X \arrow[d, "\Phi"] \\ Y \arrow[r, "H"'] & Y \end{tikzcd}$$

- Φ homeomorphism

Properties preserved by conjugation

- Sensitivity
- Expansivity
- Entropy
- ...

Strong conjugacies

For which homeomorphisms $\Phi : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is $\Phi^{-1} \circ F \circ \Phi$ a cellular automaton whenever F is a cellular automaton?

Strong conjugacies

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Theorem

When Φ is either

1. *a cellular automaton,*
2. *the reflection $\tau \colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, $\tau(x)_i := x_{-i}$ or*
3. *the composition of a cellular automaton and τ.*

Definition

Two CA are *strongly conjugate* if they are conjugate by one of (a) – (c).

Periodic points under conjugacies

Let $\mathrm{Per}_n(F)$ be the set of n-periodic points of F and let $\widetilde{\mathrm{Per}}_n(F)$ be the set of minimally n-periodic points.

$$\begin{aligned} x \in \mathrm{Per}_n(F) &\Leftrightarrow F^n(x) = x \\ &\Leftrightarrow \Phi(F^n(x)) = \Phi(x) \\ &\Leftrightarrow H^n(\Phi(x)) = \Phi(x) \\ &\Leftrightarrow \Phi(x) \in \mathrm{Per}_n(H) \end{aligned}$$

$$\Rightarrow |\mathrm{Per}_n(H)| = |\mathrm{Per}_n(F)|$$

n-periodic points vs. minimally n-periodic points

If $|\mathrm{Per}_n(F)|$ is finite for all n (e.g. for subshifts), we can reconstruct $|\widetilde{\mathrm{Per}}_n(F)|$ from $|\mathrm{Per}_n(\Phi)|$ by

$$|\widetilde{\mathrm{Per}}_n(F)| = \sum_{d|n} \mu(\frac{n}{d})|\mathrm{Per}_d(F)|.$$

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Impossible, when $|\mathrm{Per}_n(F)| \notin \mathbb{N}$, e.g. $|\mathrm{Per}_1(F)| = |\mathbb{R}|$ implies $|\mathrm{Per}_n(F)| = |\mathbb{R}|$ for all $n \in \mathbb{N}$.

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How can one determine $|\widetilde{\mathrm{Per}}_n(F)|$?

Higher block representations

Take $w_{28}(x_{-1}, x_0, x_1) = x_{-1}(1 \oplus x_1 \oplus x_0 x_1) \oplus x_0$, $n = 2$.

$x \quad \dots \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad \dots$

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$$
\begin{array}{cccccccccccccc}
x & \dots & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & \dots \\[1em]
\gamma & & & & \begin{bmatrix}1\\1\\0\\0\\1\end{bmatrix} & \begin{bmatrix}1\\0\\0\\1\\0\end{bmatrix} & \begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix} & \begin{bmatrix}0\\1\\0\\1\\0\end{bmatrix} & \begin{bmatrix}1\\0\\1\\0\\0\end{bmatrix} & \begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix} & \begin{bmatrix}1\\0\\0\\1\\1\end{bmatrix} & & &
\end{array}
$$

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x & \ldots & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & \ldots \\
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w_{28}(\gamma) & & & & \begin{bmatrix}0\\1\\0\end{bmatrix} & \begin{bmatrix}1\\0\\1\end{bmatrix} & \begin{bmatrix}0\\1\\0\end{bmatrix} & \begin{bmatrix}1\\0\\1\end{bmatrix} & \begin{bmatrix}0\\1\\1\end{bmatrix} & \begin{bmatrix}1\\1\\0\end{bmatrix} & \begin{bmatrix}1\\0\\1\end{bmatrix} & & &
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\begin{array}{ccccccccccccc}
x & \dots & 1 \quad 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \quad 1 & \dots \\
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w_{28}^2(\gamma) & & & 1 & 0 & 1 & 0 & 1 & 0 & 0 & &
\end{array}
$$

Higher block representations

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$$x \quad \dots \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad \dots$$

$$\gamma \qquad \begin{bmatrix}1\\1\\0\\0\\1\end{bmatrix} \begin{bmatrix}1\\0\\0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix} \begin{bmatrix}0\\1\\0\\1\\0\end{bmatrix} \begin{bmatrix}1\\0\\1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix} \begin{bmatrix}1\\0\\0\\1\\1\end{bmatrix}$$

$$w_{28}(\gamma) \qquad \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}1\\\mathbf{0}\\1\end{bmatrix} \begin{bmatrix}0\\\mathbf{1}\\0\end{bmatrix} \begin{bmatrix}1\\\mathbf{0}\\1\end{bmatrix} \begin{bmatrix}0\\\mathbf{1}\\1\end{bmatrix} \begin{bmatrix}1\\1\\0\end{bmatrix} \begin{bmatrix}1\\\mathbf{0}\\1\end{bmatrix}$$

$$w_{28}^2(\gamma) \qquad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0$$

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\gamma & & & & \begin{bmatrix}1\\1\\0\\0\\1\end{bmatrix} & \begin{bmatrix}1\\0\\0\\1\\0\end{bmatrix} & \begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix} & \begin{bmatrix}0\\1\\0\\1\\0\end{bmatrix} & \begin{bmatrix}1\\0\\1\\0\\0\end{bmatrix} & \begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix} & \begin{bmatrix}1\\0\\0\\1\\1\end{bmatrix} & & & \\[1em]
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w_{28}^2(\gamma) & & & & 1 & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & & &
\end{array}
$$

Periodic points as SFTs

We consider a labelled De-Bruijn-Graph G
(r is the radius of F, f the corresponding block map)

$$V(G) := \{0,1\}^{2rn},$$
$$E(G) := \{(e_1,\dots,e_{2rn}),(e_2,\dots,e_{2rn+1})\,;\,(e_1,\dots,e_{2n+1}) \in \{0,1\}^{2rn+1}\},$$
$$p(e) := \{t \in \{1,\dots,n\}\,;\, f^t(e_1,\dots,e_{2nr+1})_{(n-t)r+1} = e_{nr+1}\},$$

remove all edges where $n \notin p(e)$ and finally remove all edges not appearing in bi-infinite pathes.

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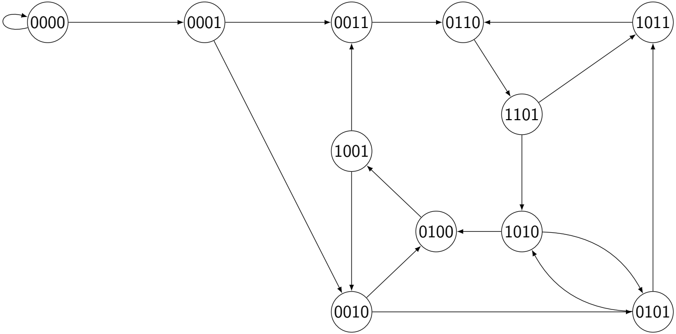
remove all edges where $n \notin p(e)$ and finally remove all edges not appearing in bi-infinite pathes. With

$$\widetilde{\mathrm{Per}}_n(G) := \{(e_i)_{i\in\mathbb{Z}} \in \mathrm{Path}(G)\ ;\ \bigcap_{i\in\mathbb{Z}} p(e_i) = \{n\}\}$$

gives us $|\widetilde{\mathrm{Per}}_n(G)| = |\widetilde{\mathrm{Per}}_n(F)|$.

The labelled De-Bruijn-Graph G for w_{28} and $n = 2$



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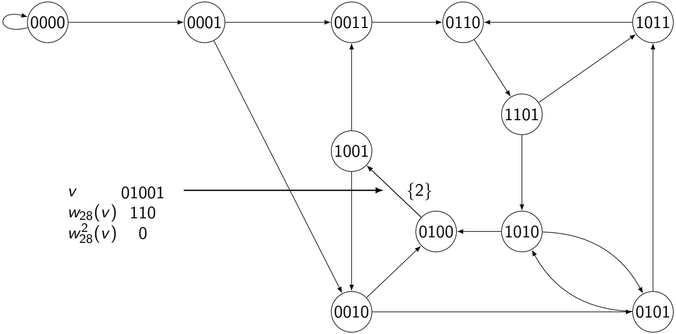
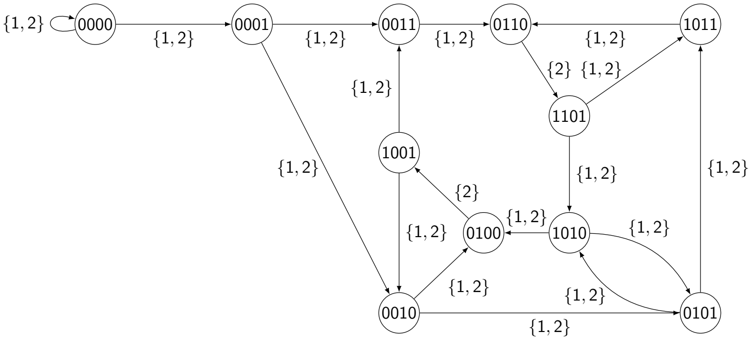


v	01001
$w_{28}(v)$	110
$w_{28}^2(v)$	0

The labelled De-Bruijn-Graph G for w_{28} and $n = 2$



The strong components digraph $S(G)$

Definition

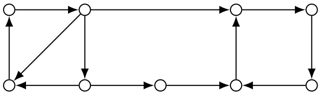
Let $S(G)$ be the graph whose vertices are the strongly connected components of G and an edge from s to t, if there is an edge in G starting in s and ending in t.

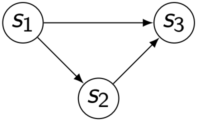
Definition

For $s_1, \ldots, s_k \in V(S(G))$ let $\mathrm{Path}(s_1, \ldots, s_k) \subseteq \mathrm{Path}(G)$ be all biinfinite pathes running through $s_1, \ldots, s_k$ in that order.

Example of $S(G)$

Example





$$|\mathrm{Path}(s_1, s_2)| = 0$$
$$|\mathrm{Path}(s_1, s_2, s_3)| = |R|$$
$$|\mathrm{Path}(s_3)| = 4$$

Substracting SFTs

Theorem

If there is a point with minimal period n in $Path(s_1, \ldots, s_k)$, then

$$|Path(s_1, \ldots, s_k) \cap \widetilde{Per}_n(G)| = |Path(s_1, \ldots, s_k)|$$

Determining $|\widetilde{\mathrm{Per}}_n(G)|$

Algorithm

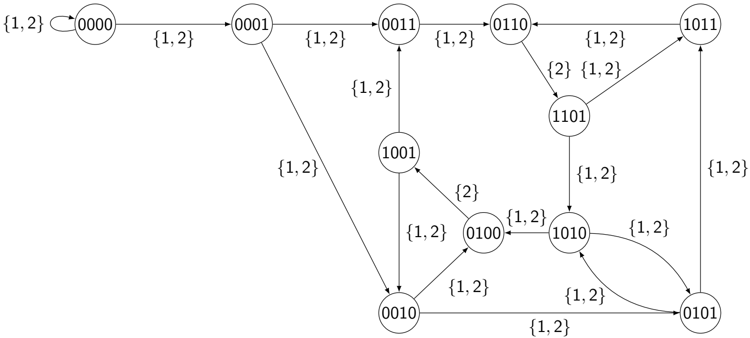
1. Decide if there is a path $\gamma \in \mathrm{Path}(s_1, \dots, s_k) \cap \widetilde{\mathrm{Per}}_n(G)$: Are there edges $s_1 \xrightarrow{e_1} s_2 \xrightarrow{e_2} \cdots \xrightarrow{e_{n-1}} s_k$ with

$$\bigcap_{i=1}^{n-1} p(e_i) \cap \bigcap_{j=1}^{n} \bigcap_{e \in E(s_i)} p(e) = \{n\}?$$

2. Determine $|\mathrm{Path}(s_1, \dots, s_k)|$ (if s_1 and s_k are not vertices):

$$\begin{aligned} |\mathrm{Path}(s_1, \dots, s_k)| = |\mathbb{R}| &\Leftrightarrow \text{one of the } s_i \text{ is not a cycle,} \\ |\mathrm{Path}(s_1, \dots, s_k)| = |\mathbb{N}| &\Leftrightarrow k \geq 2 \text{ and all } s_i \text{ are cycles,} \\ |\mathrm{Path}(s)| = \ell &\Leftrightarrow s \text{ is a cycle of length } \ell \geq 0. \end{aligned}$$

The labelled De-Bruijn-Graph G for w_{28} and $n = 2$



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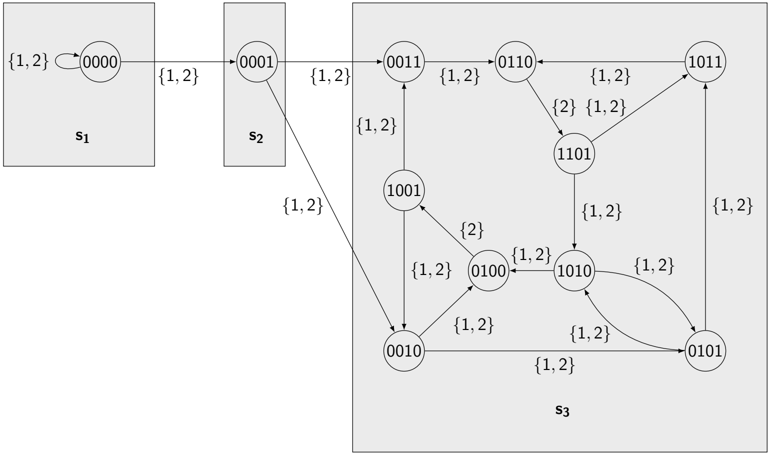

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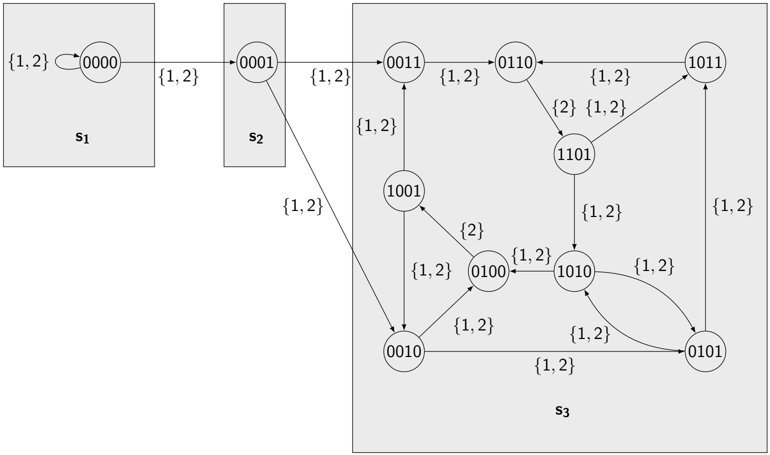
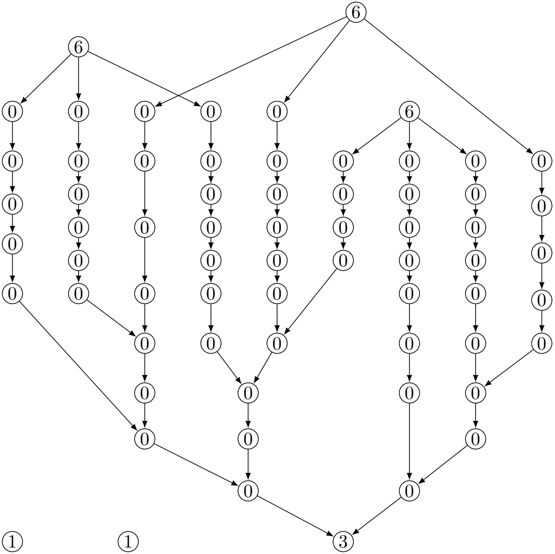


$$\mathrm{Path}(s_1, s_2) = \emptyset$$

$$\mathrm{Path}(s_1) \cap \widetilde{\mathrm{Per}}_2(f) = \emptyset$$

$$|\mathrm{Path}(s_1, s_2, s_3) \cap \widetilde{\mathrm{Per}}_2(f)| = |\mathbb{R}|$$

$S(G)$ for w_{11} and $n = 6$



Definition

Let X be a metric space and $B \subseteq X$. Define

$$\begin{aligned} D(B) &= \{x \in X \,;\, x \text{ is an accumulation point of } Y\} \\ &= \bigcap_{x \in B} \overline{B \setminus \{x\}} \end{aligned}$$

Lemma

$|D(Per_n(F))|$ is a conjugacy invariant.

Lemma

Let $\gamma \in Path(s_1, \ldots, s_k) \subseteq Path(G)$, then tfae.

- *$\gamma \notin D(Path(G))$,*
- *there is a finite subpath of γ that can not be extended to a path in G different from γ,*
- *s_1 and s_k are cycles and have no outgoing respectively incoming edges in $S(G)$.*

A big table

CA	$F^2 = id$	$F^2 = F$	$F^3 = F$	$F^4 = F^2$	$\lvert\widetilde{\mathrm{Per}}_1(F)\rvert$	$\lvert D(\mathrm{Per}_1(F))\rvert$	$\lvert D(\mathrm{Per}_1(F)^c\rvert$	$\lvert\widetilde{\mathrm{Per}}_2(F)\rvert$	$\lvert D(\mathrm{Per}_2(F))\rvert$	$\lvert D(\mathrm{Per}_2(F))^c\rvert$	$\lvert\widetilde{\mathrm{Per}}_3(F)\rvert$	$\lvert D(\mathrm{Per}_3(F))\rvert$	$\lvert D(\mathrm{Per}_3(F))^c\rvert$	$\lvert\widetilde{\mathrm{Per}}_4(F)\rvert$	$\lvert D(\mathrm{Per}_4(F))\rvert$	$\lvert D(\mathrm{Per}_4(F))^c\rvert$	$\lvert\widetilde{\mathrm{Per}}_5(F)\rvert$	$\lvert D(\mathrm{Per}_5(F))\rvert$	$\lvert D(\mathrm{Per}_5(F))^c\rvert$	$\lvert\widetilde{\mathrm{Per}}_6(F)\rvert$	$\lvert D(\mathrm{Per}_6(F))\rvert$	$\lvert D(\mathrm{Per}_6(F))^c\rvert$
0	T	T	T	T	1	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1
1	F	F	T	T	0	0	0	C	C	0	0	0	0	0	C	0	0	0	0	0	C	0
3	F	F	F	F	0	0	0	2	0	2	3	0	3	0	0	2	5	0	5	0	0	5
5	F	F	T	T	C	C	0	C	C	0	0	C	0	0	C	0	0	C	0	0	C	0
											⋮											
172	F	F	F	F	C	C	0	0	C	0	C	C	0	C	C	0	C	C	0	C	C	0
184	F	F	F	F	N	2	N	2	2	N	N	8	N	N	10	N	N	22	N	N	32	N
204	T	T	T	T	C	C	0	0	C	0	0	C	0	0	C	0	0	C	0	0	C	0
6;134	F	F	F	F	N	3	N	0	3	N	0	3	N	N	11	N	N	23	N	N	27	N
15;170	F	F	F	F	2	0	2	2	0	4	6	0	8	12	0	16	30	0	32	54	0	64
18;126	F	F	F	F	1	0	1	C	C	1	0	0	1	C	C	1	0	0	1	C	C	1
23;178	F	F	F	F	2	0	2	C	C	2	0	0	2	0	C	2	0	0	2	0	C	2
36;72	F	F	F	T	C	C	0	0	C	0	0	C	0	0	C	0	0	C	0	0	C	0
77;232	F	F	F	F	C	C	0	2	C	2	0	C	0	0	C	2	0	C	0	0	C	2
78;140	F	F	F	F	C	C	0	0	C	0	0	C	0	0	C	0	0	C	0	0	C	0
2;24;46	F	F	F	F	1	0	1	0	0	1	3	0	4	4	0	5	5	0	6	6	0	10
90;105;150	F	F	F	F	4	0	4	12	0	16	60	0	64	240	0	256	1020	0	1024	4020	0	4096
4;12;76;200	F	T	T	T	C	C	0	0	C	0	0	C	0	0	C	0	0	C	0	0	C	0

Conjugate, but not strongly conjugate

Definition

$$\nu : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{Z}, \quad \nu(x)_i = 1 - x_i.$$

Lemma

The only CA conjugate to σ are σ and σ^{-1}.

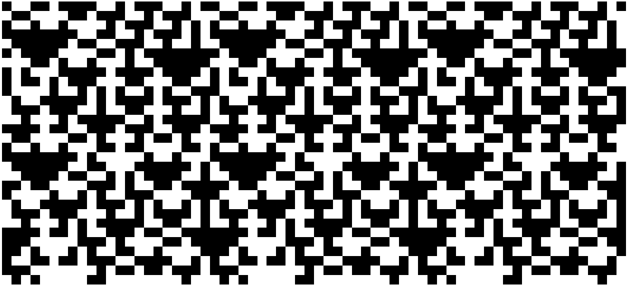
Corollary

$W_{170} = \sigma$ and $W_{15} = \sigma \circ \nu$ are conjugate, but not strongly conjugate. The conjugacy is given by flipping every second bit.

Rule 77 and 232, respectively 23 and 178 are conjugate by the same homeomorphism.

Theorem (Kurka, Nasu)

Bipermutive CA with radius r are conjugate to the one-sided full shift on $|A|^{2r}$ symbols.



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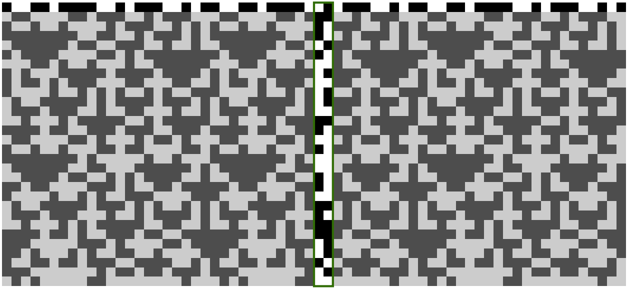

Further Invariants

The remaining CA can be distinguished by the following invariants.

- $|F^{-1}(\mathrm{Per}_1(F))|$
- $\mathrm{Fix}_k(F) := |\{x \in \mathrm{Per}_1(F)\ ;\ |F^{-1}(x)| = k\}|$
- $F(\{0,1\}^{\mathbb{Z}}) = \omega(F)$?
- $D(F^{-1}(\mathrm{Per}_1(F)))$
- $\{|F^{-1}(x)|\ ;\ x \in \{0,1\}^{\mathbb{Z}}\}$

Final Remarks

Results about the 256 elementary CA

- Exactly 83 equivalence classes up to topological conjugacy.
- Rule 4 and rule 12 are not top. conjugate but conjugate on their eventual image.
- Rule 200 and rule 76 are not top. conjugate but conjugate if we neglect the topology.

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- Rule 4 and rule 12 are not top. conjugate but conjugate on their eventual image.
- Rule 200 and rule 76 are not top. conjugate but conjugate if we neglect the topology.

Decidability

- Counting periodic points does not work in higher dimensions.
- Deciding conjugacy on the eventual image can decide nilpotency, therefore it is undecidable already in dimension one.
- Maybe deciding top. conjugacy for CA is also undecidable.