Physically Universal Cellular Automata

Luke Schaeffer

MIT

June 8, 2015

Physical Universality

Consider a CA over states Σ. Suppose we fix

- a finite set of cells X,
- a configuration y of the cells surrounding X, and
- a time $t \in \mathbb{N}$.

Physical Universality

Consider a CA over states Σ. Suppose we fix

- a finite set of cells X,
- a configuration y of the cells surrounding X, and
- a time $t \in \mathbb{N}$.

This defines a transformation,

$$f : \Sigma^X \to \Sigma^X.$$

Physical Universality

Consider a CA over states Σ. Suppose we fix

- a finite set of cells X,
- a configuration y of the cells surrounding X, and
- a time $t \in \mathbb{N}$.

This defines a transformation,

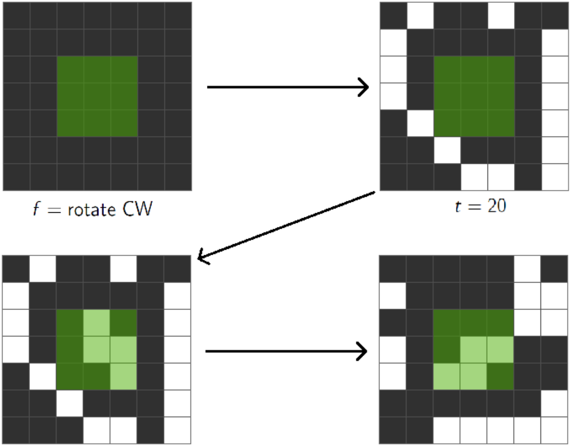
$$f : \Sigma^X \to \Sigma^X.$$

Terminology

We say the *program* (y, t) *implements* the transformation $f : \Sigma^X \to \Sigma^X$ on the *region* X.

Definition (Janzing)

A cellular automaton is *physically universal* if it can implement any transformation on any finite region.



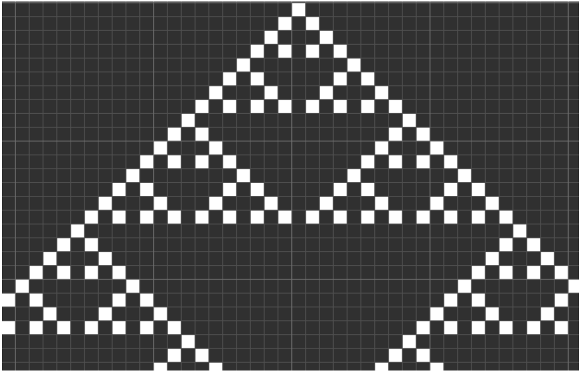
Outline

- Properties of Physically Universal CAs
  - CAs which are *not* physically universal
- A Physically Universal CA
  - Sketch argument for universality
- Reversible/Quantum Physical Universality
- Open Problems

Section 1

Properties of Physically Universal CAs

Rule 90



Property 1: Computation

Observation

A cell in rule 90 is a linear combination of the inputs.

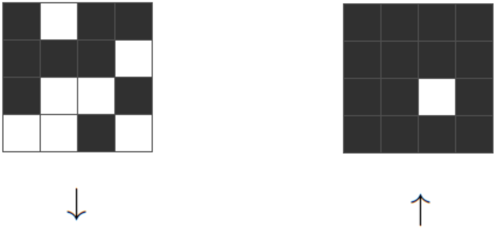
Property 1: Computation

Observation

A cell in rule 90 is a linear combination of the inputs.

Claim

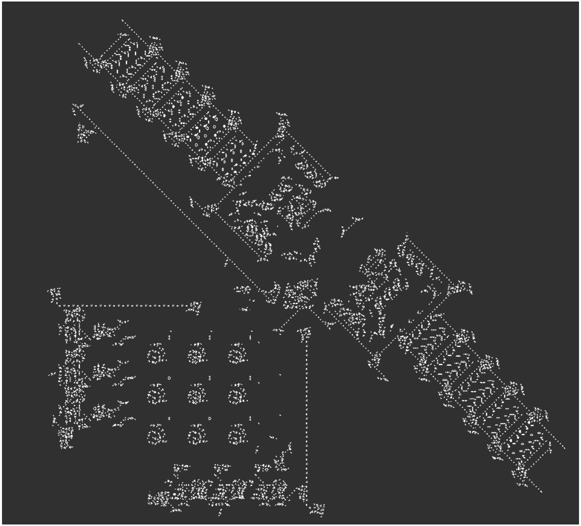
A cell in a physically universal CA can be an arbitrary function of the inputs.



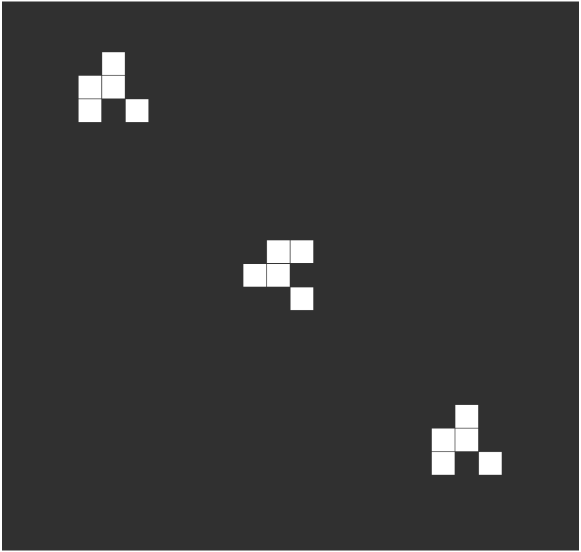
$$010\cdots1 \quad\rightarrow\quad f \quad\rightarrow\quad 1$$

Game of Life

Turing-complete.

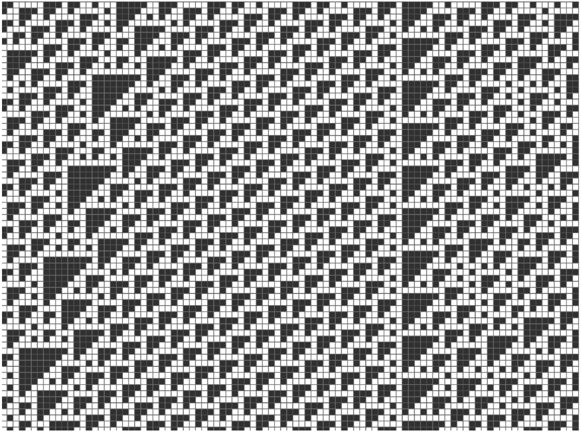


Input Tape



Rule 110

Also Turing-complete (via cyclic tag systems).



Property 2: Reversibility

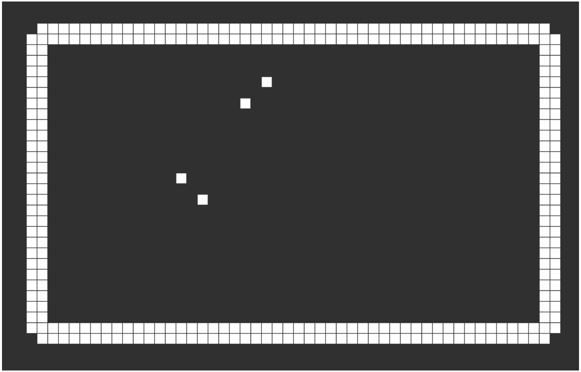
$$\begin{aligned}\text{Physical universality} &\implies \text{Injectivity} \\ &\implies \text{Reversibility}\end{aligned}$$

Property 2: Reversibility

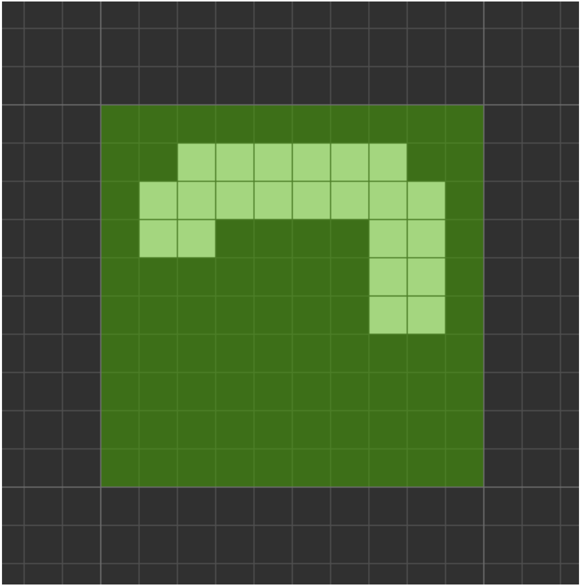
$$\text{Physical universality} \implies \text{Injectivity}$$
$$\implies \text{Reversibility}$$

- Totalistic CA are not reversible.
- Moore neighbourhood CAs are usually not reversible.
- Block cellular automata

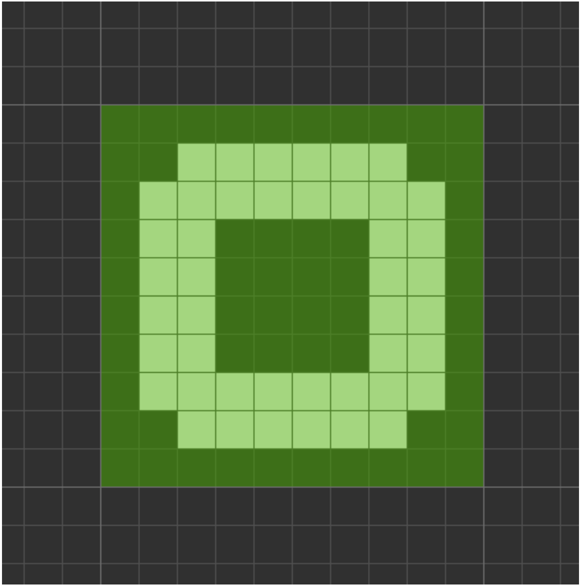
Billiard Ball Model



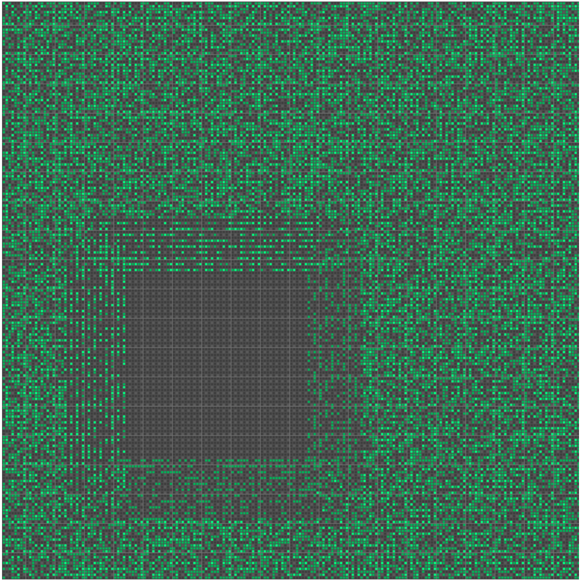
Problem 3a: Immutable cells



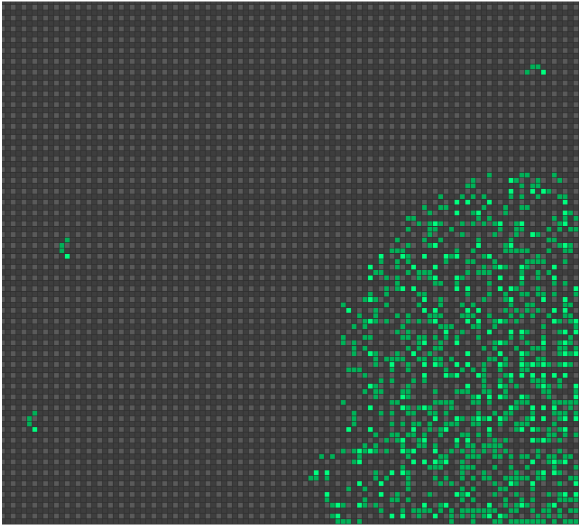
Problem 3b: Isolated systems



Hardy, Pomeau and de Pazzis gas



Critters



Section 2

A Physically Universal CA

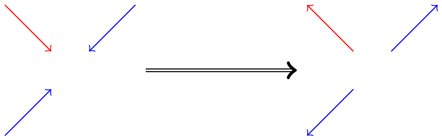
A Physically Universal CA

- Based on particles moving on a two-dimensional grid.
- Particle Properties:
  1. Each particle is at a grid point, moving in one of four directions: NE, NW, SE, SW.
  2. Particles move one cell per timestep.
  3. At most one particle with a given position/velocity.
- Particles interact at grid points.

Particle Interaction

Particle Interaction

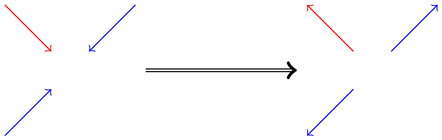
When three particles meet, the two opposing particles reflect the third particle. There is no interaction in all other cases.



Particle Interaction

Particle Interaction

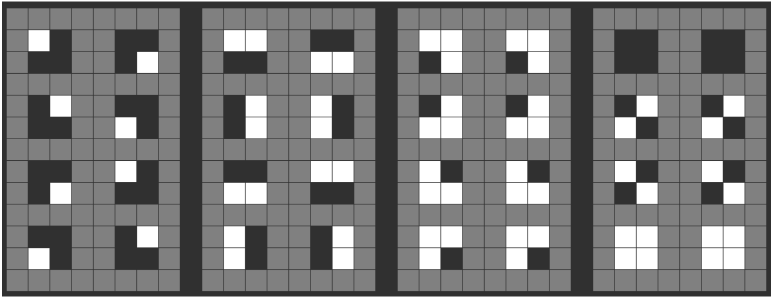
When three particles meet, the two opposing particles reflect the third particle. There is no interaction in all other cases.



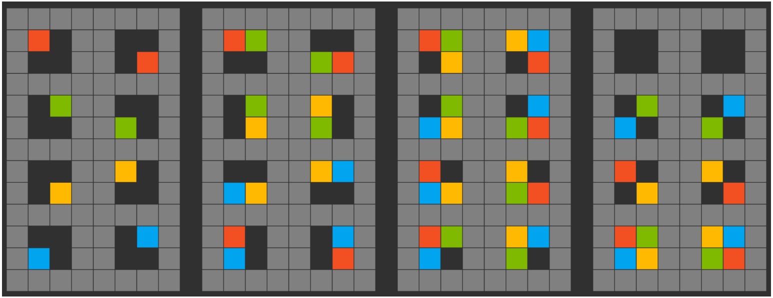
The interaction is

- reversible,
- symmetric, and
- conservative.

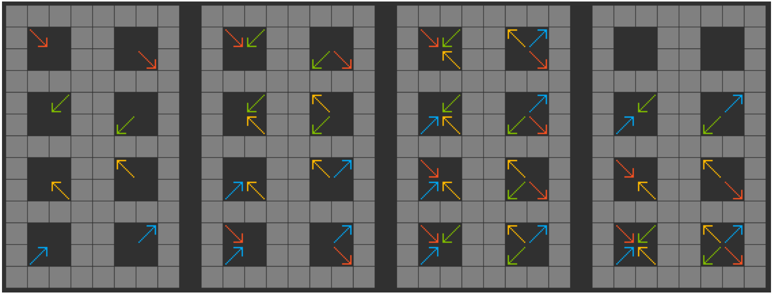
Margolus Rule



Margolus Rule



Margolus Rule



Physical Universality Checklist

Show how to

Physical Universality Checklist

Show how to

- extract information from the input region,

Physical Universality Checklist

Show how to

- extract information from the input region,
- arbitrarily manipulate the location of information,

Physical Universality Checklist

Show how to

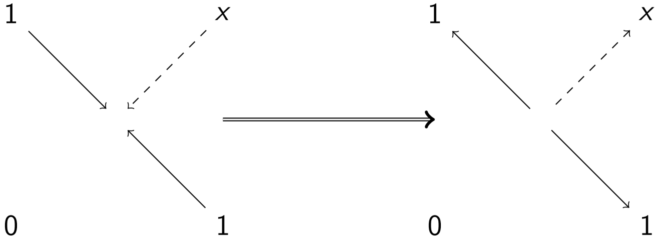
- extract information from the input region,
- arbitrarily manipulate the location of information,
- apply a universal gate, and

Physical Universality Checklist

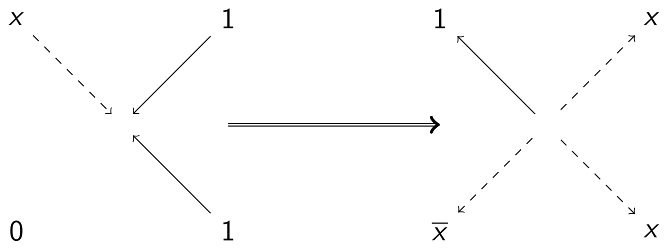
Show how to

- extract information from the input region,
- arbitrarily manipulate the location of information,
- apply a universal gate, and
- reinsert data into the region.

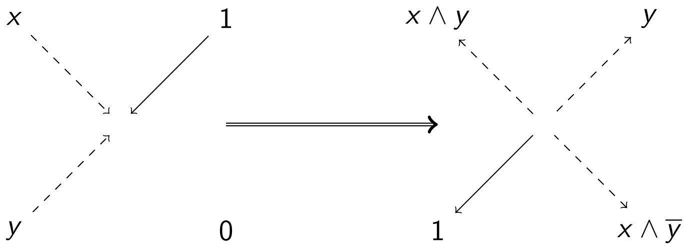
Reflection



Deflection



Computation



Physical Universality Checklist

We can

- extract information from the input region,
  - (do nothing)

Physical Universality Checklist

We can

- extract information from the input region,
  - (do nothing)
- arbitrarily manipulate the location of information,
  - (reflections and deflections)

Physical Universality Checklist

We can

- extract information from the input region,
  - (do nothing)
- arbitrarily manipulate the location of information,
  - (reflections and deflections)
- apply a universal gate, and
  - (compute $x \wedge \overline{y}$ and copy bits)

Physical Universality Checklist

We can

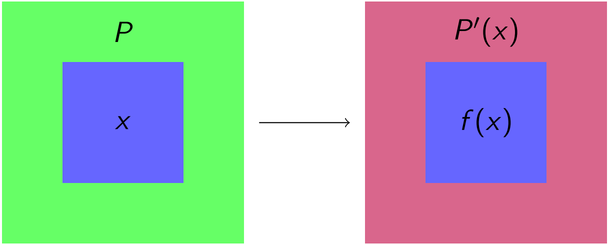
- extract information from the input region,
    - (do nothing)
- arbitrarily manipulate the location of information,
    - (reflections and deflections)
- apply a universal gate, and
    - (compute $x \wedge \overline{y}$ and copy bits)
- reinsert data into the region.
    - (follows from other steps)

Section 3

Reversible/Quantum Physical Universality

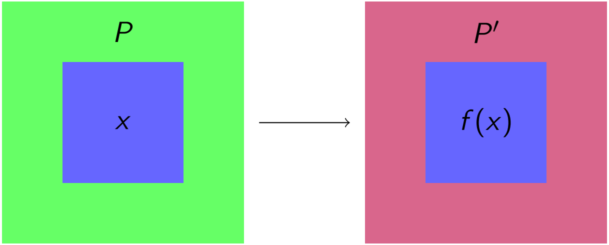
Physical Universality

Suppose f is reversible.



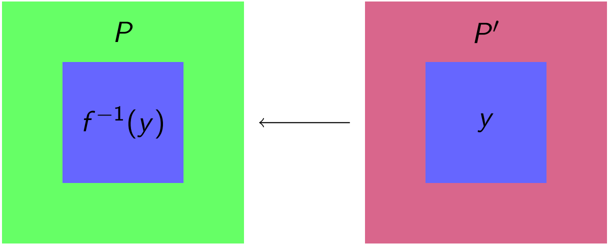
Reversible Physical Universality

If P' does not depend on x ...



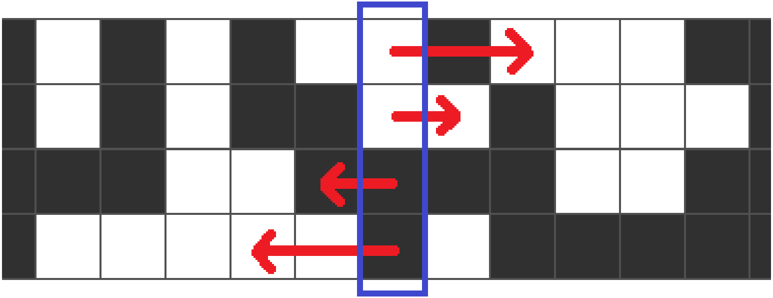
Reversible Physical Universality

. . . then P' is a program for the inverse.



Layered Cellular Automata

Use 1D layered cellular automata (Salo and Törmä)



- Each block has multiple cells, one per layer.
- Each layer moves at an integer speed.
- Alternate between transforming cells and shifting layers.

Engineered Reversibly Physically Universal CA

Build required operations into CA artificially.

Change speed If layers $1, 2, \ldots, 5$ are on then cycle layers $-5, \ldots, -1$.

Change direction If layers $-5, -4, 4, 5$ are on then swap $1, 2, 3$ with $-1, -2, -3$.

Apply reversible gate G If layers 4 and -4 are on, but 5 and -5 are off, the apply gate G to $-1, -2, -3$ and $1, 2, 3$.

Claim

With careful planning, CA will also have the diffusion property.

Physical Universality Checklist

We can

- extract information from the input region,
- arbitrarily manipulate the location of the information,
- apply a universal gate, and
- reinsert data into the region.

Quantum Cellular Automata

Analogy

Like a probabilistic cellular automaton with complex *amplitudes* instead of probabilities.

Quantum Cellular Automata

Analogy

Like a probabilistic cellular automaton with complex *amplitudes* instead of probabilities.

Instead of a distribution over outcomes, e.g.,

$$A \text{ w.p. } \frac{1}{10}$$
$$B \text{ w.p. } \frac{3}{10}$$
$$C \text{ w.p. } \frac{3}{5}$$

the state is a quantum superposition of outcomes,

$$\sqrt{\frac{1}{10}}\,|A\rangle + \sqrt{\frac{3}{10}}\,|B\rangle + \sqrt{\frac{3}{5}}\,|C\rangle$$

Probabilistic:

$$0 \rightarrow \begin{cases} 0 & \text{w.p. } \frac{2}{3} \\ 1 & \text{w.p. } \frac{1}{3} \end{cases}$$

$$1 \rightarrow \begin{cases} 0 & \text{w.p. } \frac{1}{3} \\ 1 & \text{w.p. } \frac{2}{3} \end{cases}$$

Quantum:

$$|0\rangle \rightarrow \sqrt{\frac{2}{3}}\,|0\rangle + \frac{1}{\sqrt{3}}\,|1\rangle$$

$$|1\rangle \rightarrow \frac{1}{\sqrt{3}}\,|0\rangle - \sqrt{\frac{2}{3}}\,|1\rangle$$

Probabilistic (Stochastic Matrices)

$$\frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Quantum (Unitary Matrices)

$$\frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} & 1 \\ 1 & -\sqrt{2} \end{pmatrix}$$

Quantum Physical Universality

- Quantum magic implies the following.

Quantum Physical Universality

- Quantum magic implies the following.
  - Program configurations $P_1, P_2, \ldots$ for distinct unitaries must be orthogonal.

Quantum Physical Universality

- Quantum magic implies the following.
  - Program configurations $P_1, P_2, \ldots$ for distinct unitaries must be orthogonal.
    - Program cells may as well be classical.

Quantum Physical Universality

- Quantum magic implies the following.
  - Program configurations $P_1, P_2, \ldots$ for distinct unitaries must be orthogonal.
    - Program cells may as well be classical.
  - Cannot implement all unitary operations in finite time.

Quantum Physical Universality

- Quantum magic implies the following.
    - Program configurations $P_1, P_2, \ldots$ for distinct unitaries must be orthogonal.
        - Program cells may as well be classical.
    - Cannot implement all unitary operations in finite time.
    - Instead, approximate unitary operations.

Quantum Physical Universality

- Quantum magic implies the following.
  - Program configurations $P_1, P_2, \ldots$ for distinct unitaries must be orthogonal.
    - Program cells may as well be classical.
  - Cannot implement all unitary operations in finite time.
  - Instead, approximate unitary operations.
  - It suffices to simulate quantum circuits.

Quantum Physical Universality

- Quantum magic implies the following.
  - Program configurations $P_1, P_2, \ldots$ for distinct unitaries must be orthogonal.
    - Program cells may as well be classical.
  - Cannot implement all unitary operations in finite time.
  - Instead, approximate unitary operations.
  - It suffices to simulate quantum circuits.

Already know how to simulate reversible circuits.

Section 4

Open Problems

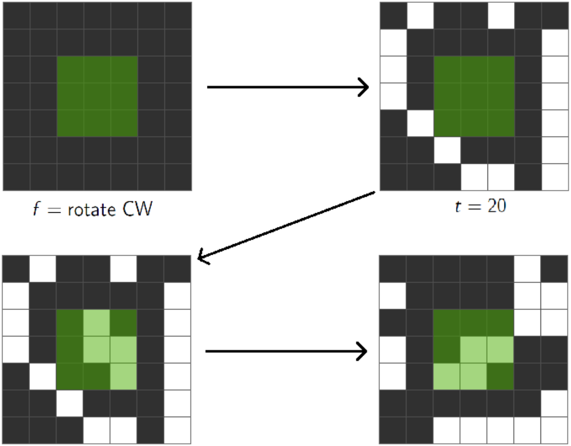
Open Problems

- Time and Space Efficiency
  - Polynomial in region size and transformation complexity
- Minimal number of states for each dimension/neighbourhood?
- Qualitatively different physically universal CAs
- Unbounded computation (e.g., Turing machines)?

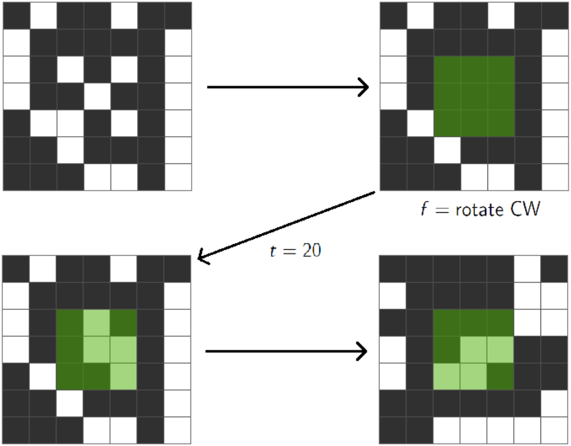
Section 5

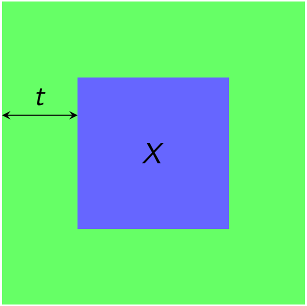
Bonus: Surprising Equivalent Definition

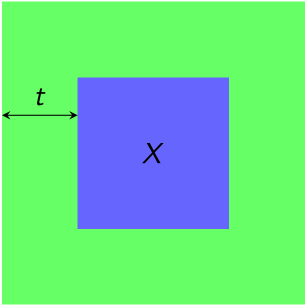
Equivalent Definition



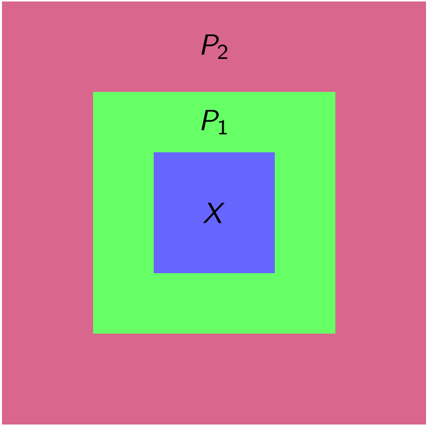
Equivalent Definition




t
X

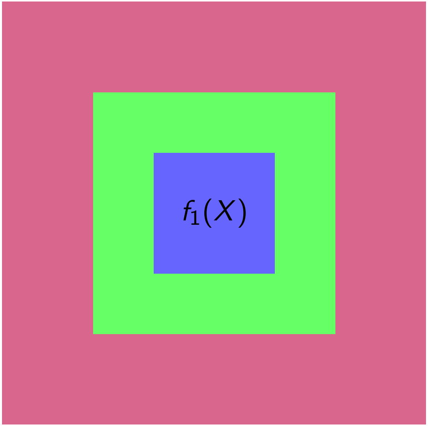


Code is data.

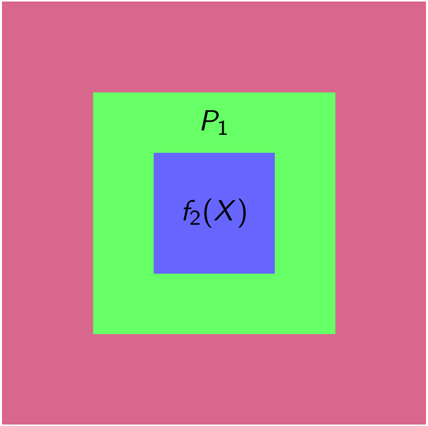


- P_1 implements f_1 on X in time t_1
- P_2 implements f_2 on X (and preserves P_1) in time t_2

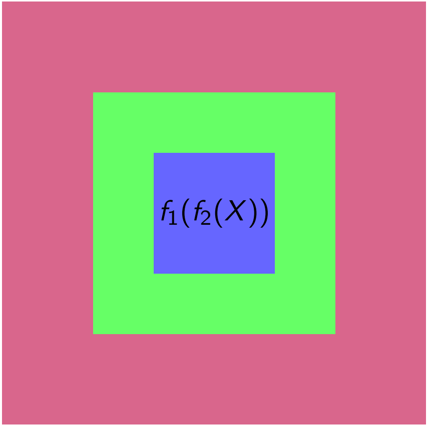
After t_1 steps:

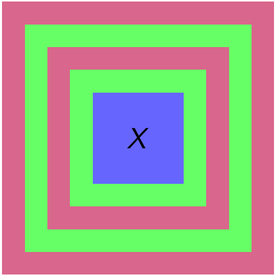


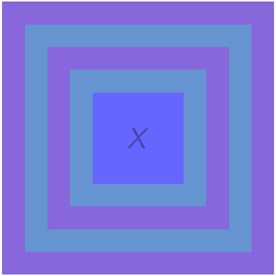
After t_2 steps:

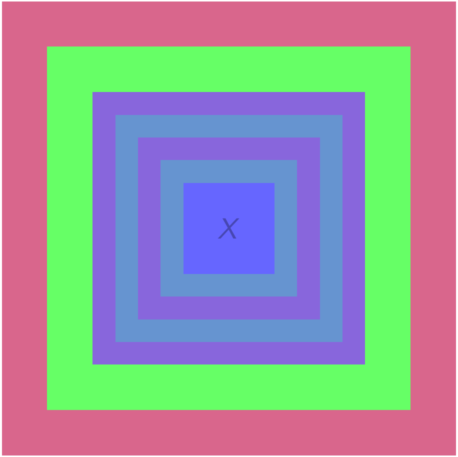


After $t_1 + t_2$ steps:




X


X


X