# INTERNATIONAL CONFERENCE ON COMPLEX ANALYSIS AND RELATED TOPICS

## The 12$^{\text{th}}$ Romanian-Finnish Seminar

August 17-21, 2009, Turku, Finland

# Program

All lectures take place at Natural Science Building II (lecture halls XX, XXIII and XXIV). The registration will take place in the conference office, Room MS-4 (between lecture halls XXIII and XXIV). The conference office is open on Sunday 18-21, Monday 8-16, Tuesday 8-16, Wednesday 8-15, Thursday 8-14, Friday 8-12.

## Monday, August 17

| | | | |
|---|---|---|---|
| 9:15-9:30 | XX | Opening Ceremony | On behalf of the Organizing Committee, Prof. Matti Vuorinen |
| 9:30-10:15 | XX | Marius Iosifescu | Iterated Function Systems and The Continued Fraction Expansion |

### Coffee

| | | | |
|---|---|---|---|
| 10:45-11:30 | XX | Lucian Beznea | Compact Superharmonic Functions and Path Regularity |

### Lunch

Session Ia: Analysis on Metric Spaces

| | | | |
|---|---|---|---|
| 14:00-14:25 | XXIII | Daniel Aalto | Maximal Operators and Sobolev Functions in Doubling Metric Measure Spaces |
| 14:30-14:55 | XXIII | Juha Lehrbäck | Pointwise Hardy Inequalities and Uniform Fatness |
| 15:00-15:25 | XXIII | Marcelina Mocanu | Orlicz-Sobolev Spaces With Zero Boundary Values on Metric Measure Spaces and Poincaré Inequalities |

Session Ib: Quasiconformal Mappings

| | | | |
|---|---|---|---|
| 14:00-14:25 | XXIV | Yury Dybov | On the Dirichlet Problem for the Beltrami Equations |
| 14:30-14:55 | XXIV | Barkat Ali Bhayo | On Mori's Theorem for Quasiconformal Maps in the $n$-space |
| 15:00-15:25 | XXIV | Riku Klén | Geometric Properties of Quasihyperbolic and $j$-metric Balls |

## Break

Session IIa: Harmonic Functions

| | | | |
|---|---|---|---|
| 16:00-16:25 | XXIII | Gheorghe Bucur | Non-Symmetric Resistance Forms |
| 16:30-16:55 | XXIII | Antti Rasila | Boundary Behavior of Harmonic and Quasiregular Mappings |

Session IIb: Classical Function Theory

| | | | |
|---|---|---|---|
| 16:00-16:25 | XXIV | Mikhail Tyaglov | The Hawaii Conjecture and Related Problems |
| 16:30-16:55 | XXIV | Ángel Alonso Gómez | Γ-lines of Algebroid Functions |

Session IIc: Topics in Analysis

| | | | |
|---|---|---|---|
| 16:00-16:25 | XX | Eugen Popa | Tensor Product of Absolutely Continuous Resolvents |
| 16:30-16:55 | XX | Kazushi Yoshitomi | Inverse Spectral Problems for Singular Rank-one Perturbations of a Hill Operator |

## 19:00 Get Together Party

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# Tuesday, August 18

| | | | |
|---|---|---|---|
| 9:00-9:45 | XX | Peter Hästö | Mappings of Finite Distortion and PDE With Nonstandard Growth |
| 10:00-10:45 | XX | Zoltan Balogh | Exceptional Sets for Absolute Continuity of Quasiconformal Maps |

## Coffee

| | | | |
|---|---|---|---|
| 11:15-12:00 | XX | Toshiyuki Sugawa | Invariant Schwarzian Derivative of Higher Order and its Applications |

## Lunch

Session Ia: Computational Methods and Applications

| | | | |
|---|---|---|---|
| 14:00-14:25 | XXIII | Harri Hakula | Computing Moduli of Rings and Quadrilaterals with *hp*-FEM |
| 14:30-14:55 | XXIII | Wolfgang Wendland | Boundary Integral Equations for Two-dimensional Low Reynolds Number Flow Past a Porous Body |

Session Ib: Univalent Functions and Classical Function Theory

| | | | |
|---|---|---|---|
| 14:00-14:25 | XXIV | Mark Elin | Boundary Behavior of Semigroups of Holomorphic Mappings |
| 14:30-14:55 | XXIV | David Shoikhet | Another Look at the Schwarz Lemma |
| 15:00-15:25 | XXIV | Petru Mocanu | Injectivity Conditions in the Complex Plane |

**Break**

Session IIa: Riemann Surfaces and Teichmüller Spaces

| | | | |
|---|---|---|---|
| 16:00-16:25 | XXIII | Ilie Barza | On the Derivatives $\partial$ and $\bar{\partial}$ on Nonorientable Klein Surfaces |
| 16:30-16:55 | XXIII | Józef Zajaç | Teichmüller Space of an Oriented Jordan Curve in the Extended Complex Plane |

Session IIb: Univalent Functions

| | | | |
|---|---|---|---|
| 16:00-16:25 | XXIV | Daniel Breaz | Some Properties for General Integral Operators |
| 16:30-16:55 | XXIV | Nicoleta Breaz | Convexity Properties for Some General Integral Operators on Uniformly Analytic Function Classes |

Session IIc: Topics in Analysis

| | | | |
|---|---|---|---|
| 16:00-16:25 | XX | Dmitry Trotsenko | Extendability of Classes of Maps and New Properties of Upper Sets |
| 16:30-16:55 | XX | Beata Fałda | Variational Pattern of the Risk Theory |

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# Wednesday, August 19

| | | | |
|---|---|---|---|
| 9:00-9:45 | XX | Alan Beardon | The Length of the Image of an Arc |
| 10:00-10:45 | XX | David Minda | Hyperbolic Distortion for Holomorphic Maps of Regions |

**Coffee**

| | | | |
|---|---|---|---|
| 11:15-12:00 | XX | Saminathan Ponnusamy | Bloch and Landau's Theorems on Harmonic and Biharmonic Mappings |

**Lunch**

Session Ia: Harmonic Functions and Quasiconformal Mappings

| | | | |
|---|---|---|---|
| 14:00-14:25 | XXIII | Abdallah Lyzzaik | Uniqueness of Harmonic Mappings into Starlike Domains |
| 14:30-14:55 | XXIII | Miodrag Mateljevic | Distortion of Quasiconformal Harmonic Functions and Harmonic Mappings |
| 15:00-15:25 | XXIII | István Prause | Quasisymmetric Distortion Spectrum |

Session Ib: Univalent Functions

| | | | |
|---|---|---|---|
| 14:00-14:25 | XXIV | Fiana Yacobzon | A Distortion Theorem for Function Convex in One Direction |
| 14:30-14:55 | XXIV | Grigore Sălăgean | Analytic Functions with Negative Coefficients and Differential Operators |
| 15:00-15:25 | XXIV | Fatima Al-Oboudi | On Classes of Functions Related to Starlike Functions with Respect to Symmetric Conjugate Points Defined by a Fractional Operator |

## Conference Dinner

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|---|---|
| ca 15:30 | Departure to Kavalto farm |
| 16:00 | Sauna |
| 19:00-21:00 | Dinner |

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# Thursday, August 20

| | | | |
|---|---|---|---|
| 9:00-9:45 | XX | Janne Heittokangas | Complex Linear Differential Equations in the Unit Disc at a Glance - an Overview of the Research Conducted on the 21st Century |
| 10:00-10:45 | XX | Gabriela Kohr | Solutions for the Generalized Loewner Differential Equation and Spirallike Mappings in $\mathbb{C}^n$ |

## Coffee

Session Ia: Classical Function Theory and Functions of Several Variables

| | | | |
|---|---|---|---|
| 11:15-11:40 | XXIII | John Pfaltzgraff | Loewner Theory and Schwarzians in $\mathbb{C}^n$ |
| 11:45-12:10 | XXIII | Andrey Osipov | On One G. Boole's Identity for Rational Functions and its Applications |

Session Ib: Topics in Analysis I

| | | | |
|---|---|---|---|
| 11:15-11:40 | XXIV | Swadesh Sahoo | Uniform Continuity and $\varphi$-uniform Domains |
| 11:45-12:10 | XXIV | Dorin Ghisa | Fundamental Domains of Riemann Zeta Function |

Session Ic: Topics in Analysis II

| | | | |
|---|---|---|---|
| 11:15-11:40 | XX | Vasile Brinzanescu | Deformations of Generalized Complex Structures on Surfaces |

## Lunch

## Social Program

| | |
|---|---|
| 14:00-16:00 | Cruise to Naantali |
| 16:00 | Arrival to Naantali |
| 18:00-21:00 | Dinner |
| 21:00 | Departure to Turku |

# Friday, August 21

| | | | |
|---|---|---|---|
| 9:00-9:45 | XX | Aurelian Gheondea | Closed Embeddings of Hilbert Spaces |

## Coffee

Session Ia: Modern Analysis

| | | | |
|---|---|---|---|
| 10:15-10:40 | XXIII | Bodo Dittmar | Maxima for the Expectation of the Lifetime of a Brownian Motion on the Ball |
| 10:45-11:10 | XXIII | Vladimir Ryazanov | On the Beltrami Equations with 2 Characteristics |

Session Ib: Topics in Analysis

| | | | |
|---|---|---|---|
| 10:15-10:40 | XXIV | Cornel Pintea | Properties of the Preimages of Monotone Operators and Some Local Concepts Which Are Equivalent to Their Global Counterparts |
| 10:45-11:10 | XXIV | Corneliu Udrea | On Generalized Laplace Equation and Nonlinear Operators |

Session Ic: Univalent Functions

| | | | |
|---|---|---|---|
| 10:15-10:40 | XX | Allu Vasudevarao | Region of Variability for Exponentially Convex Univalent Functions |
| 10:45-11:10 | XX | Árpád Baricz | Starlikeness and Convexity of Generalized Bessel Functions |

## Lunch

Session IIa: Harmonic Functions

| | | | |
|---|---|---|---|
| 13:00-13:25 | XXIII | Konstantin Fedorovskiy | Approximation by Polynomial Solutions of Elliptic Equations |
| 13:30-13:55 | XXIII | Kohur Gowrisankaran | Multiply Superharmonic Functions and Balayage Measure |
| 14:00-14:25 | XXIII | Chia-Chi Tung | On Generalized Gauss and Bochner-Martinelli Means |

Session IIb: Harmonic Functions, Riemann Surfaces and Teichmüller Spaces

| | | | |
|---|---|---|---|
| 13:00-13:25 | XXIV | Massimo Lanza de Cristoforis | Singular Perturbation Problems in Potential Theory: a Functional Analytic Approach |
| 13:30-13:55 | XXIV | Eric Schippers | Fiber Structure of the Teichmüller Space of a Bordered Riemann Surface |
| 14:00-14:25 | XXIV | Erina Kinjo | The Length Spectrum of Topologically Infinite Riemann Surfaces and the Teichmüller Metric |

# Abstracts

## MAXIMAL OPERATORS AND SOBOLEV FUNCTIONS IN DOUBLING METRIC MEASURE SPACES

**D. Aalto[1] *, J. Kinnunen[2]**

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[2]*Department of Mathematics, Helsinki University of Technology, Finland;*
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In this talk we study maximal functions of a Sobolev function in doubling metric measure space. In a Euclidean space the Hardy-Littlewood maximal operator is bounded and continuous between Sobolev spaces (see [1] and [2]). Moreover, the maximal operator does not change the boundary values of a Sobolev function defined on an open set [3].

The Sobolev spaces can be generalised in metric spaces by various different ways. We adopt the definition by Shanmugalingam [4]. We assume that the metric space is complete, supports a Poincaré inequality and that the measure is doubling. The Hardy-Littlewood maximal function of a Lipschitz function may be discontinuous even in this context (see [5]) and hence the Hardy-Littlewood maximal operator is not bounded in general. We define another maximal operator which preserves the Sobolev spaces and does not alter the boundary values. We also compare different maximal functions.

Acknowledgements. The authors were supported by the Finnish Academy of Science and Letters, the Vilho, Yrjö and Kalle Väisälä Foundation.

1. J. Kinnunen, "The Hardy-Littlewood maximal function of a Sobolev function", Israel J. Math, **100**, No. 1, 117–124 (1997).
2. H. Luiro, "Continuity of the maximal operator in Sobolev spaces", Proc. Amer. Math. Soc., **135**, No. 1, 243–251 (2007).
3. J. Kinnunen, O. Martio, "Maximal operator and superharmonicity, Function spaces", differential operators and nonlinear analysis (Pudasjärvi, 1999), Acad. Sci. Czech Repub., 157–169 (2000).
4. N. Shanmugalingam, "Newtonian spaces: an extension of Sobolev spaces to metric measure spaces", Rev. Mat. Iberoamericana 16, No. 2, 243–279 (2000).
5. S. Buckley, "Is the maximal function of a Lipschitz function continuous?", Ann. Acad. Sci. Fenn. Math., **24**, 519–528 (1999).

* represents the speaker in the case of multiple authors.

# ON CLASSES OF FUNCTIONS RELATED TO STARLIKE FUNCTIONS WITH RESPECT TO SYMMETRIC CONJUGATE POINTS DEFINED BY A FRACTIONAL OPERATOR

**F. M. Al-Oboudi**

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**Abstract.** Let $A$ be the class of analytic functions $f$ in the open unit disc $E = \{z : |z| < 1\}$, normalized by $f(0) = f'(0) - 1 = 0$. In this paper we introduce and study the class $ST_{\lambda}^{n,\alpha}(h)$ of functions $f \in A$, with $\dfrac{D_{\lambda}^{n,\alpha} f_m(z)}{z} \neq 0$, satisfying

$$\frac{z(D_{\lambda}^{n,\alpha} f(z))'}{D_{\lambda}^{n,\alpha} f_m(z)} \prec h(z), \quad z \in E,$$

where $n \in \mathbb{N}_0, 0 \leq \alpha < 1, \lambda \geq 0, m \in \mathbb{N}, h$ is a convex function in $E$, and $D_{\lambda}^{n,\alpha} : A \to A$, is the linear fractional differential operator newly defined as follows

$$D_{\lambda}^{n,\alpha} f(z) = z + \sum_{k=2}^{\infty} \Psi_{k,n}(\alpha, \lambda) a_k z^k,$$

where

$$\Psi_{k,n}(\alpha, \lambda) = \left( \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} (1 + \lambda(k-1)) \right)^n,$$

and

$$f_m(z) = \frac{1}{2m} \sum_{k=0}^{m-1} \left[ w^{-k} f(w^k z) + w^k \overline{f(w^k \overline{z})} \right], w = \exp\left(\frac{2\pi i}{m}\right).$$

For special values of the functions $h$ and the parameters $n, \alpha, m$ and $\lambda$, we get known classes of starlike functions with respect to symmetric conjugate points. Inclusion relations, convolution properties, and other results are given. Other related classes are also studied.

1. H. S. Al-Amiri, D. Coman, and P. T. Mocanu, “Some properties of starlike functions with respect to symmetric-conjugate points,” Int. J. Math. Math. Sci., **18**, No. 3, 469–474 (1995).
2. H. S. Al-Amiri, B. Green, D. Coman, and P. T. Mocanu, “Starlike and close-to-convex functions with respect to symmetric-conjugate points,” Glas. Mat., III. Ser., **30**, No. 2, 209–219 (1995).
3. F. M. Al-Oboudi, “On univalent functions defined by a generalized Salagean operator,” Int. J. Math. Math. Sci., **2004**, No. 27, 1429–1436 (2004).
4. F. M. Al-Oboudi, and K. A. Al-Amoudi, “On classes of analytic functions related to conic domains,” J. Math. Anal. Appl., **339**, No. 1, 655–667 (2008).
5. R. Chand, and P. Singh, “On certain schlicht mappings,” Indian J. Pure Appl. Math., **10**, No. 9, 1167–1174 (1979).

6. R. N. Das and P. Singh, "On subclasses of schlicht mapping," Indian J. Pure Appl. Math., **8**, No. 8, 864–872 (1977).
7. Rabha Md. El-Ashwah, and D. K. Thomas, "Some subclasses of close-to-convex functions," J. Ramanujan Math. Soc., **2**, No. 1, 85–100 (1987).
8. P. T. Mocanu, "On starlike functions with respect to symmetric points," Bull. Math. Soc. Sci. Math. Roum., Nouv. Sér., **28**, No. 1, 47–50 (1984).
9. P. T. Mocanu, "Certain classes of starlike functions with respect to symmetric points," Mathematica, **32**, No. 55, 153–157 (1990).
10. S. Owa and H. M. Srivastava, "Univalent and starlike generalized hypergeometric functions," Can. J. Math., **39**, No. 5, 1057–1077 (1987).
11. St. Ruscheweyh, Convolutions in Geometric Function Theory, Sem. Math. Sup. 83, Presses Univ. de Montreal (1982).
12. St. Ruscheweyh, and T. Sheil-Small, "Hadamard products of schlicht functions and the Polya-Schoenberg conjecture," Comment. Math. Helv., **48**, 119–135 (1973).
13. K. Sakaguchi, "On certain univalent mappings," J. Math. Soc. Japan, **11**, 72–75 (1959).
14. G. S. Salagean, "Subclasses of univalent functions. ," Complex Analysis - Proc. 5th Rom.-Finn. Semin., Bucharest 1981, Part 1, Lect. Notes Math., **1013**, 362–372 (1983).
15. T. N. Shanmugam, C. Ramachandran, and V. Ravichandran, "Fekete-Szegao problem for subclasses of starlike functions with respect to symmetric points," Bull. Korean Math. Soc., **43**, 589–598 (2006).

## Γ-LINES OF ALGEBROID FUNCTIONS

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[2] *Universidad Nacional de Educación a Distancia, Madrid, Spain;*
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In this lecture several theorems of the theory of Γ-lines for meromorphic functions are extended to the more general setting of algebroid functions. We recall the definition of algebroid function of order k and how it can be considered as a function defined on a Riemann surface of k sheets. In this way, we prove the so called tangent variation principle for algebroid functions, previously proved for meromorphic funtions by G. Barsegian, and we get several consequences of this result. We also extend a proposition on proximity properties of meromorphic functions.

1. G. Barsegian, "A Proximity Property of the $a$-Points of Meromorphic Functions", Math. URSS Sbornik Vol. 48 No. 1, pp. 41-63 (1984).
2. G. Barsegian, Γ-lines: On the Geometry of Real and Complex Functions, Taylor & Francis, London (2002).

## EXCEPTIONAL SETS FOR ABSOLUTE CONTINUITY OF QUASICONFORMAL MAPS

**Z. Balogh**

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One of the basic results of Euclidean quasiconformal mappings is the absolute continuity property on almost every line. This property has been generalized to qc maps on the Heisenberg group and in more general metric spaces. We shall study the size of lines which are exceptional for this property both in the Euclidean and in more general metric settings.

## STARLIKENESS AND CONVEXITY OF GENERALIZED BESSEL FUNCTIONS

**Á. Baricz**

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In this lecture we give suffcient conditions for the parameters of the normalized form of the generalized Bessel functions to be convex and starlike in the open unit disk. As an application of our main results we solve a recent open problem concerning a subordination property of Bessel functions with different parameters. Moreover, we present a new inequality for the Euler gamma function, which we apply in order to have tight bounds for the generalized and normalized Bessel function of the first kind.

The talk is based on a recent work with S. Ponnusamy.

## ON THE DERIVATIVES $\partial$ and $\bar{\partial}$ ON NONORIENTABLE KLEIN SURFACES

**I. Barza[1] *, D. Ghisa[2]**

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Let $\mathbf{X}$ be a *nonorientable* Klein surface and $\mathcal{O}_2 = \mathcal{O}_2(\mathbf{X})$ be its orientable double cover, which is a *symmetric* Riemann surface in the meaning of Klein. This means that there exists an antianalytic involution *without fixed points* $\mathbf{h} : \mathcal{O}_2 \to \mathcal{O}_2$ such that $\mathbf{X}$ is dianalitically equivalent with the orbit space $\mathcal{O}_2/ < \mathbf{h} >$, $< \mathbf{h} >$ being the group (with two elements) generated by the involution $\mathbf{h}$.

The theme of our communication is to show the $\mathbf{h}_*$-invariant vector fields on $\mathcal{O}_2$ which correspond to the derivatives $\partial$ and $\bar{\partial}$ on the surface $\mathbf{X}$.

1. I. Barza and D. Ghisa, “Vector Fields on Nonorientable Surfaces”, International Journal of Mathematics and Mathematical Sciences, Volume 2003, No. 3, pp. 133–152 (2003).

## THE LENGTH OF THE IMAGE OF AN ARC

**A. F. Beardon**

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Let $L(r)$ be the Euclidean length of the image under an analytic map of the radial segment $[0, r]$ in the unit disc. We reprove some of the classical estimates for $L(r)$ (when $L(r)$ is unbounded) in terms of hyperbolic geometry, and then use these ideas to motivate new questions, some of which we answer.

## COMPACT SUPERHARMONIC FUNCTIONS AND PATH REGULARITY

**L. Beznea**

*"Simion Stoilow" Institute of Mathematics of the Romanian Academy, Bucharest, Romania;*
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We discuss the relations between the existence of the $\mathcal{L}$-superharmonic functions that have compact level sets ($\mathcal{L}$ being the generator of a right Markov process), the path regularity of the process, and the tightness of the induced capacities. We present several examples, mainly in infinite dimensional situations, like the case when $\mathcal{L}$ is the Gross-Laplace operator on an abstract Wiener space. We deduce the càdlàg property of the paths of a class of measure-valued branching process associated with nonlinear operators of the form $\mathcal{L}u + \Phi(u)$, where $\Phi$ is a "branching mechanism" (in particular we may take $\Phi(u) = -u^{\alpha}$, with $1 < \alpha \leq 2$), completing results of P.J. Fitzsimmons and E.B. Dynkin.

# ON MORI'S THEOREM FOR QUASICONFORMAL MAPS IN THE $n$-SPACE

**B.A. Bhayo**

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R. Fehlmann and M. Vuorinen [4] proved in 1988 that Mori's constant $M(n, K)$ for $K$-quasiconformal maps of the unit ball $\mathbf{B}^n$ onto itself keeping the origin fixed satisfies $M(n, K) \to 1$ when $K \to 1$. We give here an alternative proof of this fact, with a quantitative upper bound for the constant in terms of elementary functions. Our proof is based on a refinement of a method due to G.D. Anderson and M. K. Vamanamurthy [1]. We also give an explicit version of the Schwarz lemma for quasiconformal self-maps of the unit disk. Some experimental data is presented to estimate the Mori constant when the dimension is 2. This talk is based on the paper [3], which is part of the author's PhD thesis, currently in preparation under the supervision of Prof. Matti Vuorinen.
Acknowledgements. The author was supported by the Graduate School of Mathematical Analysis and its Applications .

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3. B. A. Bhayo and M. Vuorinen, "On Mori's theorem for quasiconformal maps in the $n$-space", arXiv:0906.5853v1 [math.CA] (16 Jun 2009).
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# SOME PROPERTIES FOR GENERAL INTEGRAL OPERATORS

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For analytic functions $f_j(z)$ in the open unit disk $\mathbb{U}$ with $f_j(0) = 0$ and $f_j'(0) = 1$, two general integral operators $F_\beta(z)$ and $G_\beta(z)$ are introduced. In view of the results due to S. Owa, J. Nishiwaki and N. Niwa (Int. J. Open Problems Compt. Math. **1**(2008), 1 - 7), new classes $\mathcal{T}_\delta^*(\alpha), \mathcal{S}_\delta^*(\alpha), \mathcal{K}_\delta^*(\alpha)$, and $\mathcal{C}_\delta^*(\alpha)$ are considered. The object of the present paper is to discuss some properties for the general integral operators $F_\beta(z)$ and $G_\beta(z)$ with the above classes.

The main result of this talk is the following theorem.

**Theorem 1.** *If $f_j(z) \in \mathcal{S}_{\delta_j}^*(\alpha_j)$ for each $j = 1, 2, 3, \cdots, n$, then*

$$\operatorname{Re}\left(\tfrac{zF_\beta''(z)}{F_\beta'(z)}\right) < \tfrac{(1-\alpha)\beta}{2\delta(\alpha+1)} \quad (z \in \mathbb{U}), \tag{2.3}$$

*where*

$$\frac{1-\alpha}{2\delta(\alpha+1)} = \max_{1 \leqq j \leqq n} \frac{1-\alpha_j}{2\delta_j(\alpha_j+1)}$$

*and $\sum_{j=1}^n \beta_j = \beta$. This implies that $F_\beta(z) \in \mathcal{S}_{\frac{\delta}{\beta}}^*(\alpha)$.*

**Theorem 2.** *If $f_j(z) \in \mathcal{T}_{\delta_j}^*(\alpha_j)$ for each $j = 1, 2, 3, \cdots, n$, then*

$$\operatorname{Re}\left(\tfrac{zF_\beta''(z)}{F_\beta'(z)}\right) > \tfrac{(\alpha-1)\beta}{2\delta(\alpha+1)} \quad (z \in \mathbb{U}), \tag{2.4}$$

*where*

$$\frac{\alpha-1}{2\delta(\alpha+1)} = \min_{1 \leqq j \leqq n} \frac{\alpha_j-1}{2\delta_j(\alpha_j+1)}$$

*and $\beta = \sum_{j=1}^n \beta_j$. This implies that $F_\beta(z) \in \mathcal{T}_{\frac{\delta}{\beta}}^*(\alpha)$.*

**Theorem 3.** *If $f_j(z) \in \mathcal{K}_{\delta_j}^*(\alpha_j)$ for each $j = 1, 2, 3, \cdots, n$, then*

$$\operatorname{Re}\left(\tfrac{zG_\beta''(z)}{G_\beta'(z)}\right) < \tfrac{(1-\alpha)\beta}{2\delta(\alpha+1)} \quad (z \in \mathbb{U}), \tag{3.4}$$

*where*

$$\frac{1-\alpha}{2\delta(\alpha+1)} = \max_{1 \leqq j \leqq n} \frac{1-\alpha_j}{2\delta_j(\alpha_j+1)}$$

*and $\beta = \sum_{j=1}^n \beta_j$. This means that $G_\beta(z) \in \mathcal{S}_{\frac{\delta}{\beta}}^*(\alpha)$.*

**Theorem 4.** *If $f_j(z) \in \mathcal{C}_{\delta_j}^*(\alpha_j)$ for each $j = 1, 2, 3, \cdots, n$, then*

$$\operatorname{Re}\left(\tfrac{zG_\beta''(z)}{G_\beta'(z)}\right) > \tfrac{(\alpha-1)\beta}{2\delta(\alpha+1)} \quad (z \in \mathbb{U}), \tag{3.5}$$

*where*

$$\frac{\alpha-1}{2\delta(\alpha+1)} = \min_{1 \leqq j \leqq n} \frac{\alpha_j-1}{2\delta_j(\alpha_j+1)}$$

*and $\beta = \sum_{j=1}^n \beta_j$. This means that $G_\beta(z) \in \mathcal{T}_{\frac{\delta}{\beta}}^*(\alpha)$.*

1. D. Breaz, S. Owa and N. Breaz, "A new general integral operator", Acta Universitatis Apulensis, **16**, 11–16 (2008).
2. P.L. Duren, Univalent Functions, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo (1983).
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# CONVEXITY PROPERTIES FOR SOME GENERAL INTEGRAL OPERATORS ON UNIFORMLY ANALYTIC FUNCTION CLASSES

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In this paper the authors prove some properties for two general integral operators on the classes $\beta - \mathcal{UCV}(\alpha)$ and $\beta - \mathcal{S}_p(\alpha)$.

We consider the next general integral operators defined by

$$F_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z) = \int_0^z (f_1'(t)))^{\gamma_1} \cdot \ldots \cdot (f_n'(t))^{\gamma_n} dt$$

with $f_i \in \mathcal{A}$, $\gamma_i > 0$, $i = 1, 2, \ldots, n$ and

$$G_\gamma(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\gamma_1} \cdot \ldots \cdot \left(\frac{f_n(t)}{t}\right)^{\gamma_n} dt$$

with $f_j \in \mathcal{A}$, $\gamma_j > 0$, $j = 1, 2, \ldots, n$.

The main result of this talk is the following theorems.

**Theorem 2.1.** Let $f_i \in \beta_i - \mathcal{UCV}(\alpha_i)$, for all $i \in \{1, \ldots, n\}$, for $-1 \le \alpha_i \le 1$. Then $F_{\gamma_1,\gamma_2,\ldots,\gamma_n} \in \mathcal{K}(\rho)$, where $\rho = 1 + \sum_{i=1}^{n} \gamma_i(\alpha_i - 1)$.

**Theorem 2.4.** Let $f_i \in \beta_i - \mathcal{S}_p(\alpha_i)$, for all $i \in \{1, \ldots, n\}$, for $-1 \le \alpha_i \le 1$. Then $G_\gamma \in \mathcal{K}(\delta)$, where $\delta = 1 + \sum_{i=1}^{n} \gamma_i(\alpha_i - 1)$.

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## DEFORMATIONS OF GENERALIZED COMPLEX STRUCTURES ON SURFACES

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We compute the deformations in the sense of generalized complex structures of the standard classical complex structure on a primary Kodaira surface and we prove that the obtained family of deformations is a smooth locally complete family depending on four complex parameters. This family is the same as the extended deformations (in the sense of Kontsevich and Barannikov) in degree two, obtained by Poon using differential Gerstenhaber algebras. We also compute the deformations in the sense of generalized complex structures of the standard complex structure on a complex 2-torus. We get a smooth complete family depending on six complex parameters and, in particular we obtain the well-known smooth complete family of complex deformations depending on four parameters.

## NON-SYMMETRIC RESISTANCE FORMS

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Resistance symmetric forms represent a special case of Dirichlet forms on the space of $L^2$ functions with respect to a sigma-finite measure. The present state of the theory is the one presented in the monograph J. Kigami: Analysis on Fractals. Cambridge Tracts in Mathematics, 2001. Such Dirichlet forms had as a starting point the classical papers of A. Beurling and J. Deny: Espaces de Dirichlet. I. Le

cas elementaire. Acta. Math. **99** 1958, 203 - 224, and Dirichlet spaces, Proc. Nat. Acad. Sci. U.S.A. **45** 1959, 208 - 215. Subsequent developments, especially the study of Markov processes associated to these Dirichlet forms have been brought by M. Fukushima: Dirichlet forms and Markov processes, North-Holland Mathematical Library, 23, 1980. and M. L. Silverstein: Symmetric Markov Processes, Lecture Notes in Mathematics, 426, Springer, 1974. Very soon a theory of nonsymmetric Dirichlet forms has been developed, in strong connection with a series of models from the theory of elliptic differential operators, see Z.M. Ma, M. Roeckner: Introduction to the Theory of (Non-Symmetric) Dirichlet forms, Springer Universitext 1992.

Gh. Bucur is planning to study non-symmetric resistance forms, in order to obtain a theory that will represent the "resistance alternative" to non-symmetric Dirichlet forms. Starting with a given set one will develop a potential theory associated to a resistance form defined on the complement of a finite set that will play the role of space boundary. There will be studied excessive functions with respect to this resistance form, functions that form a cone which in the classical case corresponds to the family of superharmonic functions.

# GEOMETRIC PROPERTIES OF OPEN, DISCRETE MAPPINGS SATISFYING A GENERALIZED MODULAR INEQUALITY

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We continue our work from [1], [2] of studying the geometric properties of a class of open, discrete mappings for which a generalized Poleckii's modular inequality holds. The class of mappings from [1] and [2] extends some classes of mappings of finite distortion with the dilatation in the $BMO$ class or so that $e^{AK_0(f)}$ is locally integrable for some Orlicz map $A$. We introduce now a class of open, discrete mappings which satisfies an extended form of Poleckii's modular inequality. We show that a map $f$ from this class satisfies condition $(N)$ and we study the boundary extention properties, equicontinuity and eliminability properties.

1. M. Cristea, Local homeomorphisms having local $ACL^n$ inverses, Complex Variables and Elliptic Equations, 1, 2008, 77-99.
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# SINGULAR PERTURBATION PROBLEMS IN POTENTIAL THEORY: A FUNCTIONAL ANALYTIC APPROACH

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This talk is dedicated to the analysis of boundary value problems on singularly perturbed domains by an approach which is alternative to those of asymptotic analysis and of homogenization theory.

In particular, we will consider a certain linear or nonlinear boundary value problem on a domain with one or possibly infinitely many holes, whose size is determined by a positive parameter $\epsilon$ and we will consider a family of solutions depending on $\epsilon$ as $\epsilon$ approaches 0. Then we shall represent the dependence on $\epsilon$ of the family of solutions, or of corresponding functionals of the solutions such as the energy integral, in terms of possibly singular at 0 but known functions of $\epsilon$ such as $\epsilon^{-1}$ or $\log \epsilon$, and in terms of possibly unknown real analytic operators.

# MAXIMA FOR THE EXPECTATION OF THE LIFETIME OF A BROWNIAN MOTION ON THE BALL

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Let $G_B(x,y)$ be the Green's function of a domain $B$ and $\Gamma_B(x,y) = \int_B G_B(x,z)G_B(z,y)dz$ the iterated Green's function, which is one of the Green's functions of the biharmonic equation. The expectation of the lifetime of a Brownian motion in $B$, starting in $x$, conditioned to converge to and to be stopped at $y$ and to be killed on exiting $B$ is known to be equal

$$E_x^y = \frac{\Gamma_B(x,y)}{G_B(x,y)}.$$

It is of interest to know where the expectation for the lifetime $E_x^y$ attained maxima. The talk deals with the case of the disk in the plane and the ball in the space. Basing on [1] it is proven in the case of the disk using only elementary conformal mapping that $E_x^y$ cannot obtain its maximum at interior points the same is proven for the case of the ball if at least one point is on the boundary.

1. B. Dittmar, “Local and global maxima for the expectation of the lifetime of a Brownian motion on the disk”, J. Analyse, **104**, 59 - 68 (2008).

# VECTOR BOUNDARY RIEMANN PROBLEMS ON THE "INFINITE CHAIN" OF THE COMPACT SURFACES

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Let the infinite "chain" of the algebraic surfaces be given and each of them is the covering of the preceding one. Every monodromy group that raises every next surface from the chain is assumed to be commutative and the respective vector boundary Riemann problem has the commutative permutation matrix coefficient.

It is proved that all interim subcoverings as the uniform covering of the whole "chain" too have the noncommutative monodromy groups and correspond to the appropriate vector boundary Riemann problems with the noncommutative permutation matrix coefficients. These problems are solved by means of the simpler one.

The above mentioned facts are important and interesting not only from the aspect of the classical complex analysis but also because of their applications to the various parts of the current mathematical physics.

All present results are considerable generalization of [1], [2] and are obtained explicitly.

1. I. Dmitrieva, "On some generalizations of the classical Riemann problem", in: Proc. of the ICIAM07, 16-20.07.07, ETHZ, Zürich, Switzerland, ZAMM (special issue, electronic version), Germany, 2007, (publication process).
2. I. Dmitrieva, "Vector boundary Riemann problems on the compact surfaces and their applications in the algebraic function theory", in: Proc. of the XIth Romanian-Finnish Seminar, Int. Conf on Complex Anal. and Related Topics, 14-19.08.08, Alba-Iulia University, Romania (under the review process)

# DIFFERENTIAL EQUATION FOR THE DISTURBED ISOTROPIC MEDIUM IN THE CASE OF THE COMPLEX SET OF THE TRANSFORMATION PARAMETERS

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The concrete industrial problems were solved in [1] and they dealt with the study of the signal transmissions in the various kinds of media.

Mathematically these problems were considered as the generalized Maxwell system of PDEs that was diagonalized successfully [2]. The final diagonalization step was expressed as the uniform scalar PDE of the fourth order in the case of the disturbed isotropic medium over the classical Maxwell space. This equation was reduced to the ODE of the same order with respect to the time parameter. The reduction was done by means of the appropriate integral transformations that were applied to the usual spatial three-dimensional real coordinates.

The above mentioned ODE was solved explicitly and the most interesting aspect of its study dealt with the complex set of the integral transformation parameters.

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2. I. Yu. Dmitrieva, "On the constructive solution of the n-dimensional differential operator equations' system and its application to the classical Maxwell theory", in: Proc. of the 6th Congress of Romananian Mathematicians, 28.06-4.07.07, IMAR, Bucharest, Romania, p.p. 231-236. (publication process)

# ON THE LINDELÖF THEOREM

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Let $D$ be a bounded domain in $\mathbb{C}^n$, $n > 1$, with $C^2$-smooth boundary $\partial D$, $T_\xi(\partial D)$ be the tangent space to $\partial D$ at $\xi \in \partial D$, and let $\nu_\xi$ be the unit outward normal vector to $\partial D$ at $\xi$. Let $k_D$ be the infinitesimal Kobayashi metric, and let $K_D$ be the Kobayashi metric on $D$. Denote by $\mathbb{C}\nu_\xi$ the complex the complex normal space. Let $\pi_\xi : \mathbb{C}^n \to \mathbb{C}\nu_\xi$ be the orthogonal projection.

An open set $\mathcal{K} \subset D$ is called weakly admissible domain at $\xi \in \partial D$ if $\mathcal{K}$ is asymptotic at $\xi$, $\mathcal{K} \cap \mathbb{C}\nu_\xi \subset \Gamma$, where $\Gamma$ is an angle in $D \cap \mathbb{C}\nu_\xi$, with vertex $\xi$, and $\lim_{\mathcal{K} \ni z \to \xi} K_D(z, \pi_\xi(z)) = 0$.

The following example help to explain weakly admissible domain (for the proof see [2])

$$A^\epsilon_\alpha(\xi) = \{\, z \in D \ : |(z-\xi, \nu_\xi)| < (1+\alpha)\delta_\xi(z), |z-\xi|^2 < \alpha\delta_\xi^{1+\epsilon}(z)\,\},$$
$$\delta_\xi(z) = \min\{p[z, \partial D], p[z, T_\xi(\partial D)]\}.$$

Here $\alpha, \varepsilon > 0$, $(\cdot,\cdot)$ denotes canonical hermitian product of $\mathbb{C}^n$, and $p$ denotes the Euclidean distance in $\mathbb{C}^n$.

If $\xi \in \partial D$, $f : D \to \overline{\mathbb{C}}$, $l \in \overline{\mathbb{C}}$, we say $f$ has weakly admissible limit $l$ at $\xi$ if $\lim_{\mathcal{K} \ni z \to \xi} f(z) = l$ for every weakly admissible domain $\mathcal{K}$ at $\xi$.

We shall say that a curve $\gamma : [0,1) \to D$ ending at $\xi \in \partial D$ is *special* at $\xi$ if $\gamma_\xi = \pi_\xi \circ \gamma$ is a Jordan arc lying in $D \cap \mathbb{C}\nu_\xi$, except for the point $\xi$, and $\lim_{t \to 1-} K_D(\gamma(t), \pi_\xi \circ \gamma(t)) = 0$.

Let $H^\infty(D)$ be the set of all bounded holomorphic functions on $D$.

The main result of this talk is the following theorem.

**Theorem 1.** *Let $D$ be a bounded domain in $\mathbb{C}^n$ with $C^2$-smooth boundary, $f \in H^\infty(D)$, and let $\gamma_1$ and $\gamma_2$ be two special curves lying in $D$, and terminating at a point $\xi$ of $\partial D$. If $Ref(\lambda) \to a$ as $\lambda \to \xi$ on $\gamma_1$ and $Imf(\lambda) \to b$ as $z \to \xi$ on $\gamma_2$ then $f$ then $f$ has weakly admissible limit $a + ib$ at $\xi$.*

This result is a generalization of a well-known classical theorem of Gehring and Lohwater [1]. The Lindelöf–Lehto–Virtanen theorem for normal holomorphic functions of several complex variables was established in [2].

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# ON THE DIRICHLET PROBLEM FOR THE BELTRAMI EQUATIONS

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Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The equation

$$f_{\overline{z}} = \mu\,(z)\, f_z \tag{1}$$

where $z = x + iy$, $f_{\bar{z}} = (f_x + if_y)/2$, $f_z = (f_x - if_y)/2$ and $|\mu(z)| < 1$ a.e., is said to be a **Beltrami equation.** Boundary problems for the Beltrami equation with $\mu \equiv 0$ are due to the famous dissertation of Riemann and to the known works of Hilbert (1904, 1924) and Poincare (1910) and, for the uniformly elliptic case, with $||\mu||_\infty < 1$ to Vekua.

Given a continuous function $\varphi : \partial\mathbb{D} \to \mathbb{R}$, $\varphi(\zeta) \neq const$, a discrete open function $f : \mathbb{D} \to \mathbb{C}$ of the class $W^{1,1}_{loc}$ is called a **regular solution** of the Dirichlet problem for (1) if $f$ satisfies (1) a.e. and $\lim\limits_{z\to\zeta} \operatorname{Re} f(z) = \varphi(\zeta) \quad \forall\, \zeta \in \partial\mathbb{D}$, $\operatorname{Im} f(0) = 0$ and its Jacobian $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 \neq 0$ a.e. Under $\varphi(\zeta) \equiv c$, $\zeta \in \partial\mathbb{D}$, the regular solutions is $f(z) \equiv c$, $z \in \mathbb{D}$.

**Theorem 1.** *Let $\mu : \mathbb{D} \to \mathbb{D}$ be a measurable function such that*

$$K_\mu(z) := \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \;\leq\; Q(z) \in BMO(\mathbb{D})\,. \tag{2}$$

*Then the Beltrami equation (1) has a regular solution of the Dirichlet problem for every continuous function $\varphi : \partial\mathbb{D} \to \mathbb{R}$.*

Here **BMO** stands to the class of functions of **bounded mean oscillation** by John and Nirenberg and **VMO** to the class of **vanishing mean oscillation** by Sarvas. By the Brezis-Nirenberg theorem $W^{1,2}(\mathbb{D}) \subset VMO(\mathbb{D})$ and since $VMO(\mathbb{D}) \subset BMO(\mathbb{D})$ we have the following.

**Corollary 1** *Let $\mu : \mathbb{D} \to \mathbb{D}$ be a measurable function such that*

$$K_\mu(z) \;\leq\; Q(z) \;\in\; W^{1,2}_{loc}(\mathbb{D})\;. \tag{3}$$

*Then the conclusion of the Theorem 1 holds.*

**Remark 1.** Furthemore, it can be shown on the base of the result of Martio-Miklyukov that in the latter case $f \in W^{1,2}_{loc}$.

Acknowledgements. I'd like to thank Professor Vladimir Ryazanov for setting of the problem and his supervision of my research.

# BOUNDARY BEHAVIOR OF SEMIGROUPS OF HOLOMORPHIC MAPPINGS

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The asymptotic behavior of semigroups of holomorphic mappings and their boundary behavior attract considerable attention of many mathematicians through a long period but especially in the last decade.

In this talk we discuss some recent quantative characteristics of boundary asymptotic behavior of such semigroups acting on the open unit disk of the complex plane. In particular, we present new results on the limit curvature of semigroups trajectories at the boundary Denjoy–Wolff point.

The talk is based on a joint work with D. Shoikhet.

# VARIATIONAL PATTERN OF THE RISK THEORY

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Making use of the theory concerning variational calculus we propose in this presentation a new risk measure constructed on the base of the square variational norm. Basic properties, examples and applications to the stock market analysis will be presented during this lecture.

# APPROXIMATION BY POLYNOMIAL SOLUTIONS OF ELLIPTIC EQUATIONS

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Let $L$ be a homogenous elliptic differential operator in $\mathbb{C}$ with constant complex coefficients. A function $f$ is called $L$-analytic on an open set $U \subset \mathbb{C}$ if $Lf = 0$ in $U$; a polynomial $P$ is called $L$-polynomial if $LP \equiv 0$. Let $X$ be a compact set in $\mathbb{C}$, $X^\circ$ be its interior and $m \geq 1$ be integer. We consider the following problems:

1. What conditions on $X$ are necessary and sufficient in order that each function which is continuous on $X$ and $L$-analytic on $X^\circ$ can be uniformly on $X$ approximated by $L$-polynomials, and

2. What conditions on $X$ are necessary and sufficient in order that each function $f$ which is of the class $C^m$ in a neighbourhood of $X$ and $L$-analytic on $X^\circ$ can be approximated by some sequence $(P_n)$ of $L$-polynomials so that $P_n \to f$ and $\nabla^k P_n \to \nabla^k f$ uniformly on $X$ as $n \to \infty$ and $k = 1, \dots, m$?

In the talk it is planned to discuss the state of the art, some recent results (see [1,2,3,4,5]) and open questions in these themes; the special attention will be given to the case when $L = \bar{\partial}^n$ is the $n$-th power of the Cauchy-Riemann operator (in this case one deals with approximation by *polyanalytic polynomials*).

ACKNOWLEDGEMENTS. The author is supported by the program "Leading Scientific Schools of the Russian Federation" (project no. NSh-3877.2008.1).

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# CLOSED EMBEDDINGS OF HILBERT SPACES

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We introduce the notion of closely embedded Hilbert spaces, as generalizations of operator ranges and continuously embedded Hilbert spaces. These spaces are associated to unbounded selfadjoint operators that play the role of kernel operators, and they are special representations of induced Hilbert spaces. Certain canonical representations and characterizations of uniqueness are obtained. In terms of kernel operators, this corresponds to noncummutative absolute continuity and Radon-Nikodym derivatives. We exemplify these constructions by certain Hilbert spaces associted to multiplication operators, to Hilbert spaces of holomorphic functions, and some singular integral operators.

# FUNDAMENTAL DOMAINS OF RIEMANN ZETA FUNCTION

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Let $f : D \to \widehat{C}$ be a meromorphic function, where $D \subset \widehat{C}$ is the existence domain of $f$. We call fundamental domain of $f$ any simply connected domain $\Omega \subseteq D$ such that $\Omega$ is mapped conformally by $f$ onto the complex plane with a slit.

We prove that $D$ can be written as a disjoint union of sets whose interior are fundamental domains.

If $f$ is rational of degree $n$, the number of fundamental domains is exactly $n$. For transcendental functions the number of fundamental domains is infinite and they accumulate to every essential singularity of $f$ and only there. We deal in this paper with the fundamental domains of Riemann Zeta function and show how their study allows one to tackle the celebrated Riemann hypothesis regarding the non trivial zeros of that function.

# INTEGRALLY QUASICONFORMAL MAPPINGS IN SPACE

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We consider the class of mappings whose dilatations are finite in a certain integral sense. The features of such mappings are close to usual quasiconformal mappings. However, there are also some essential differences. For example, it is well known that the all classical dilatations of quasiconformality are finite or infinite simultaneously; but this is not true for their integral counterparts. Another difference concerns the important differential property of classical quasiconformal mappings: being from the Sobolev space $W^{1,n}_{\mathrm{loc}}$, they necessarily must belong to $W^{1,n+\varepsilon}_{\mathrm{loc}}$ for some $\varepsilon > 0$ (Gehring, Reshetnyak), but this fails for mappings with integrally bounded dilatations. In this talk we establish various geometric properties of generalized quasiconformal mappings, the distortion estimates and related results.

# PROOF OF GENERAL UNIFORMISATION THEOREM FOR PLANAR RIEMANN SURFACES AND ITS APPLICATIONS

**V.V. Hemasundar Gollakota**

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The object of the talk is to present a transparent and complete proof of Koebe's General Uniformisation Theorem which asserts that "Every planar Riemann surface is biholomorphic to a domain in Riemann sphere". We also indicate how Koebe's Theorem can be used to construct compact Riemann surfaces of every genus (explicitly in case of g = 1) in a very concrete way.

We show here how, for a relatively compact domain with good boundary, the planarity condition just means that the boundary curves generate the homology of the domain. In the sequel we use this result along with the beautiful construction of Weyl [3] to prove that the boundary curves form an *integral* basis for the homology of the domain. That is the periods of any smooth closed one-form on the domain are the *integral* linear combination of the integrals of the form along the boundary curves.

Finally we show that the method of proof used in [1] (to prove that plane domains of finite connectivity with analytic boundary are biholomorphic to circular-slit annuli) can be carried out for domains with analytic boundary on planar Riemann surface. However, our proof for the injectivity of the constructed mapping function seems to be new, and we feel it is more satisfactory than the one given in [1].

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# MULTIPLY SUPERHARMONIC FUNCTIONS AND BALAYAGE MEASURE

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Balayage measures in Harmonic spaces are well known and play an important role. In this talk we shall introduce a notion of balayage measures in product of Harmonic spaces and consider some of the consequences.

# COMPUTING MODULI OF RINGS AND QUADRILATERALS WITH *hp*-FEM

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In this talk we demonstrate the effectiveness of the *hp*-version of the finite element method (FEM)[1] in computing moduli of rings and quadrilaterals. In two dimensions the minimization of the Dirichlet integral

$$\int_{\Omega} |\nabla u|^2 dm$$

can be interpreted as minimization of the square of the potential energy. This is exactly the quantity that the FEM minimizes. The singularities associated with the corners of the domains and interfaces of Dirichlet and Neumann boundary conditions are resolved using a combination of geo- metric grading of the meshes and high-order polynomial basis functions.

In many cases it is possible to obtain accurate results down to machine- level accuracy (double precision). We review a set of numerical experi- ments and discuss them in relation to known benchmark results [2,3].

This work has been done in close collaboration with Antti Rasila and prof Matti Vuorinen [4].

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# MAPPINGS OF FINITE DISTORTION AND PDE WITH NONSTANDARD GROWTH

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Bojarski and Iwaniec ($p > 2$) and Manfredi ($1 < p < \infty$) showed that that if $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ and $u \in W^{1,p}_{loc}$ is non-constant, then $\partial_z u = \frac{1}{2}(u_x - iu_y)$ is $K_p$-quasiregular, with

$$K_p = \frac{1}{2}\Big(p - 1 + \frac{1}{p-1}\Big).$$

In other words, the gradient of every $p$-harmonic function is $K_p$-quasiregular.

In recent years both $p$-harmonic functions and $K$-quasiregular mappings have been extended to include the case where the parameter $p$ or $K$ depends on the space variable, leading to $p(\cdot)$-harmonic functions and mappings of finite distortion $K(x)$ (also called below $K(\cdot)$-quasiregular); In this talk I present results by T. Adamowicz and myself on extending the Bojarski–Iwaniec–Manfredi result to this setting.

An obvious conjecture would be that the gradient of a $p(\cdot)$-harmonic function is $K_p(\cdot)$-finite distortion with

$$K_p(x) = \frac{1}{2}\left(p(x) - 1 + \frac{1}{p(x)-1}\right).$$

Unfortunately, the relationship in the variable exponent case is not quite as simple as this; in fact, for arbitrarily regular $p$, say $p \in C^\infty(\overline{\Omega})$, the gradient of a $p(\cdot)$-harmonic function need not be of finite distortion at all.

It turns out that the central question is how we generalize the $p$-Laplace equation to the variable exponent, non-standard growth setting. The usual formulation starting from the weak form does not give solutions with the desirable geometric properties that we are interested in, such as being of finite distortion.

The correct starting point is to look at the strong form of the $p$-Laplace equation $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$, i.e.

$$\Delta_p u(x) = |\nabla u|^{p-4}\left[(p-2)\sum_{i,j} u_{x_i,x_j} u_{x_i} u_{x_j} + |\nabla u|^2 \Delta u\right] = 0,$$

where $u_{x_i}$ denotes the partial derivative. If $p$ is replaced by $p(x)$, we arrive at another generalization of the $p$-Laplace equation, which is denoted by $\widetilde{\Delta}_{p(\cdot)} u$.

Since our starting point is the strong form of the $q$-Laplace equation, it is natural to use this as leverage in order to prove existence and regularity results for solutions. This is the approach adopted by us. Our main result in the case $1 < p^- \leqslant p^+ < \infty$ is the following generalization of the Bojarski–Iwaniec–Manfredi result:

**Theorem.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded $C^2$ domain and let $g \in C^{1,\gamma}(\partial\Omega)$. Suppose that $p$ is Lipschitz continuous and that $1 < p^- \leqslant p^+ < \infty$. Then there exists a bounded weak solution $u \in C^{1,\gamma}(\overline{\Omega})$ of*

$$\begin{cases} \widetilde{\Delta}_{p(\cdot)} u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

*which satisfies the strong maximum principle*

$$\sup_D |u| \leqslant \sup_{\partial D} |u|$$

*for every $D \subset \Omega$. Moreover, the complex gradient $\frac{1}{2}(u_x - iu_y)$ of $u$ is $K_p(\cdot)$-quasiregular with*

$$K_p(x) = \frac{1}{2}\left(p(x) - 1 + \frac{1}{p(x) - 1}\right).$$

This result gives us good control locally of the mapping properties of $F$, for instance, if $p = 2$ in some open set, then $F$ is conformal in this set. But we also note that the condition $1 < p^- \leqslant p^+ < \infty$ implies that $K_p(\cdot) \in L^\infty$, so in fact our mapping is quasiregular. However, using more advanced techniques, it is also possible to prove a corresponding result for the case $p^- = 1$, when the gradient thus is shown to be of finite distortion proper.

The presentation is based on joint work with Tomasz Adamowicz (University of Cincinnati)

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# COMPLEX LINEAR DIFFERENTIAL EQUATIONS IN THE UNIT DISC AT A GLANCE - AN OVERVIEW OF THE RESEARCH CONDUCTED ON THE 21ST CENTURY

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We will begin with a short overview of some results on complex differential equations in the unit disc written on the 20th Century. This will lead us to the main part of the talk, which is devoted to considering the development of the research work conducted on the 21st Century. The talk contains an overview of some of the main results in about a dozen of papers where the speaker has been either an author or a co-author. Some related research by other authors are pointed out also.

A large portion of the 21st Century research is based on a result by H. Wittich published in 1966: *The entire coefficients of*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f'(z) + A_0(z) f = 0$$

*are polynomials if and only if all solutions are entire functions of finite order of growth.* In the unit disc analogue the polynomial coefficients are replaced with analytic functions $A_j(z)$ having the growth rate

$$\sup_{z\in D}(1-|z|^2)^{\alpha_j}|A_j(z)| < \infty,$$

where the constants $\alpha_j \geq 0$ correspond to the degrees of polynomials.

ACKNOWLEDGEMENTS.The author was supported by the Academy of Finland #210245 and #121281, and the Väisälä Fund of the Finnish Academy of Science and Letters.

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# ITERATED FUNCTION SYSTEMS AND THE CONTINUED FRACTION EXPANSION

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Let $\Omega$ be the set of irrationals in $I = [0,1]$ and let $[a_1(\omega), a_2(\omega), \cdots]$ denote the continued fraction expansion of $\omega \in \Omega$. The sequence $(a_n)_{n\in\mathbf{N}_+}$, $\mathbf{N}_+ = \{1, 2, \dots\}$, is defined a.e. in $I$ and is strictly stationary on the probability space $(I, \mathcal{B}_I, \gamma)$, where $\mathcal{B}_I$ is the collection of Borel subsets of $I$ and $\gamma$ is Gauss' measure on $\mathcal{B}_I$ defined by

$$\gamma(A) = \frac{1}{\log 2}\int_A \frac{\mathrm{d}x}{x+1},\ A \in \mathcal{B}_I.$$

Cf. ([2], Section 1.2).

Define extended incomplete quotients $\overline{a}_\ell,\ \ell \in \mathbf{Z} = \{\cdots, -1, 0, 1, \cdots\}$, by $\overline{a}_\ell(\omega,\theta) = a_\ell(\omega), \overline{a}_0(\omega,\theta) = a_1(\theta), \overline{a}_{-\ell}(\omega,\theta) = a_{\ell+1}(\theta)$ for $\ell \in \mathbf{N}_+$ and $(\omega,\theta) = \Omega^2$. The doubly infinite sequence $(\overline{a}_\ell)_{\ell\in\mathbf{Z}}$ is defined a.e. in $I^2$ and is strictly stationary on the probability space $(I^2, \mathcal{B}_{I^2}, \overline{\gamma})$, where $\mathcal{B}_{I^2}$ is the collection of Borel subsets of $I^2$ and $\overline{\gamma}$ is the extended Gauss measure on $\mathcal{B}_{I^2}$ defined by

$$\overline{\gamma}(B) = \frac{1}{\log 2}\int\int_B \frac{\mathrm{d}x\mathrm{d}y}{(xy+1)^2},\ B \in \mathcal{B}_{I^2}.$$

Cf. ([2], Subsections 1.3.1 and 1.3.3).

Set $\overline{s}_\ell = [\overline{a}_\ell, , \overline{a}_{\ell-1}, \ldots],\quad \ell \in \mathbf{Z}$, and $s_0^a = a,\ s_{n+1}^a = 1/(a_{n+1} + s_n^a),\ a \in I,\ n \in \mathbf{N} = \{0\} \cup \mathbf{N}_+$. We prove that for any $a \in I$ the sequence $(\ s_n^a,\ s_{n+1}^a, \ldots)$ on $(I, \mathcal{B}_I, \gamma)$ converges in distribution as $n \to \infty$ to the strictly stationary sequence $(\overline{s}_0, \overline{s}_1 \cdots)$ on $(I^2, \mathcal{B}_{I^2}, \overline{\gamma})$. In particular,

$$\lim_{n\to\infty} \gamma(s_n^a < x) = \gamma([0,x]),\ x \in I,$$

for any $a \in I$, cf. ([2], Subsection 2.5.3), and

$$\lim_{n\to\infty}\gamma\left(s_n^a < x,\ s_{n+1}^a < y\right) = \lim_{n\to\infty}\overline{\gamma}(\overline{s}_0 < x,\ \overline{s}_1 < y) = \lim_{n\to\infty}\overline{\gamma}(\overline{s}_{-1} < x,\ \overline{s}_0 < y)$$

$$= \begin{cases} \dfrac{1}{\log 2}\log\left(1 + \dfrac{x}{[1/y]+1}\right) & \text{if } x \le \{1/y\} \\[2ex] \dfrac{1}{\log 2}\log\dfrac{(y+1)([1/y]+x)}{[1/y]+1} & \text{if } x > \{1/y\} \end{cases}$$

for any $a \in I$ and $(x,y) \in I^2$. Here, $[\cdot]$ and $\{\cdot\}$ stand for whole part and fractionary part, respectively.

In the proof we use Kingman's subadditive ergodic theorem (see [3]) and work of J.H. Elton [1] for iterated function systems with Lipschitz self-mappings obeying a strictly stationary mechanism instead of an i.i.d. one.


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# THE LENGTH SPECTRUM OF TOPOLOGICALLY INFINITE RIEMANN SURFACES AND THE TEICHMÜLLER METRIC


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Let $R$ be a hyperbolic Riemann surface. We consider two metrics on the Teichmüller space $T(R)$. In 1972, T. Sorvali [3] defined a metric $d_L$ on $T(R)$ by the length spectrum of Riemann surfaces of $T(R)$. He asked the following problem: Does the metric $d_L$ define the same topology as that of the Teichmüller metric $d_T$ on $T(R)$ if $R$ is a topologically finite Riemann surface? In 1999, Liu gave a positive answer to this question. In 2003, Shiga [2] showed that there exists a Riemann surface $R$ of infinite type such that $d_L$ on $T(R)$ does not define the same topology as that of $d_T$. Also, he gave a sufficient condition for these metrics to have the same topology on $T(R)$. In 2008, Liu, Sun, Wei [1] obtained a necessary condition by showing that for some topologically infinite Riemann surfaces which do not satisfy Shiga's sufficient condition, $d_T$ and $d_L$ do not define the same topology on the Teichmüller spaces.

In this talk, we show that there exist topologically infinite Riemann surfaces which do not satisfy Shiga's sufficient condition such that $d_T$ and $d_L$ define the same topology on the Teichmüller spaces.

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# GEOMETRIC PROPERTIES OF QUASIHYPERBOLIC AND $j$-METRIC BALLS

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We will consider geometric properties, such as convexity and starlikeness, of the metric balls defined by the quasihyperbolic metric and the $j$-metric. We will introduce some new results concerning close-to-convexity. We will show that the $j$-metric balls with small radii are close-to-convex in general subdomains and the quasihyperbolic balls with small radii are close-to-convex in the punctured space. The following theorem is our main result.

**Theorem 1.** *For a domain $G \subsetneq \mathbb{R}^n$ and $x \in G$ the $j$-metric ball $B_j(x,r)$ is close-to-convex, if $r \in (0, \log(1+\sqrt{3})]$.*
*For $y \in \mathbb{R}^n \setminus \{0\}$ the quasihyperbolic ball $B_k(y,r)$ is close-to-convex, if $r \in (0, \lambda]$,*

*where* $\lambda$ *has a numerical approximation* $\lambda \approx 2.97169$.
*Moreover, the constants* $\log(1+\sqrt{3})$ *and* $\lambda$ *are sharp in the case* $n = 2$.


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# SOLUTIONS FOR THE GENERALIZED LOEWNER DIFFERENTIAL EQUATION AND SPIRALLIKE MAPPINGS IN $\mathbb{C}^n$


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In this talk we present a survey of up-to-date results in the theory of Loewner chains and the generalized Loewner differential equation on the unit ball $B^n$ in $\mathbb{C}^n$. We are mainly interested in the form of arbitrary solutions of the Loewner differential equation. In particular, we determine the form of the univalent solutions. The results are applied to subordination chains generated by spirallike mappings on $B^n$. Finally, we determine the form of solutions in the presence of certain coefficient bounds. Various applications and generalizations to $L^d$-Loewner chains will also be discussed.


Acknowledgements. The present talk is based on joint work with Mirela Kohr (Cluj - Napoca), Ian Graham (Toronto), Peter Duren (Ann Arbor), and Hidetaka Hamada (Fukuoka).

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## POINTWISE HARDY INEQUALITIES AND UNIFORM FATNESS

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We consider the following result for open sets in the Euclidean space $\mathbb{R}^n$:

**Theorem 1.** *Let $\Omega \subsetneq \mathbb{R}^n$ be an open set. Then the following conditions are quantitatively equivalent:*
*(a) The complement $\Omega^c = \mathbb{R}^n \setminus \Omega$ is uniformly p-fat, that is,*

$$\mathrm{cap}_p(\Omega^c \cap \overline{B}(w,r), B(w,2r)) \geq C\,\mathrm{cap}_p(\overline{B}(w,r), B(w,2r))$$

*for all $w \in \Omega^c$ and all $r > 0$, where $\mathrm{cap}_p$ is the variational p-capacity.*
*(b) For all $u \in C_0^\infty(\Omega)$, the pointwise p-Hardy inequality*

$$|u(x)|^p \leq C d(x,\partial\Omega)^p M_{2d(x,\partial\Omega)}(|\nabla u|)^p$$

*holds at every $x \in \Omega$; here $M_R$ denotes the restricted Hardy–Litlewood maximal operator.*

Note that by the maximal theorem, the above pointwise $p$-Hardy inequality implies the usual integral version of the Hardy inequality for all $q > p$:

$$\int_\Omega \frac{|u(x)|^q}{d(x,\partial\Omega)^q}\,dx \leq C \int_\Omega |\nabla u(x)|^q\,dx.$$

We indicate how to obtain this also for $q = p$. In addition, we consider related Poincaré- and Hausdorff content -type boundary conditions.

Parts of this talk are from a joined work with Riikka Korte and Heli Tuominen, and the results hold in rather general metric spaces as well.

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# UNIQUENESS OF HARMONIC MAPPINGS INTO STARLIKE DOMAINS

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Let $\Omega$ be a bounded simply connected domain containing a point $\omega_0$ and having a locally connected boundary, and let $\omega$ be an analytic function of the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ satisfying $|\omega| < 1$. It is known that there exists a one-to-one planar harmonic map of $\mathbb{D}$ normalized by $f(0) = \omega_0$ and $f_z(0) > 0$ and which maps $\mathbb{D}$ into $\Omega$ such that (i) the unrestricted limit $f^*(e^{it}) = \lim_{z\to e^{it}} f(z)$ exists and belongs to $\partial\Omega$ for all but a countable subset $E$ of the unit circle $\mathbb{T} = \partial\mathbb{D}$, (ii) $f^*$ is a continuous function on $\mathbb{T} \setminus E$ such that for every $e^{is} \in E$ the one-sided limits $\lim_{t\to s^+} f^*(e^{it})$ and $\lim_{t\to s^-} f^*(e^{it})$ exist, belong to $\partial\Omega$, and are distinct, and (iii) the cluster set of $f$ at $e^{is} \in E$ is the straight line segment joining the one-sided limits $\lim_{t\to s^+} f^*(e^{it})$ and $\lim_{t\to s^-} f^*(e^{it})$. The authors have recently established that this solution is unique if $\Omega$ is a strictly starlike domain with respect to $\omega_0$ with a rectifiable boundary. In this paper it is shown that the latter result also holds under the same assumptions but for the weaker assumption that $\Omega$ is only a starlike domain with respect to $\omega_0$.

# DISTORTION OF QUASICONFORMAL HARMONIC FUNCTIONS AND HARMONIC MAPPINGS

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Harmonic quasiconformal mappings were first studied by O. Martio. We estimate the modulus of derivatives of mappings which satisfy a certain estimate concerning laplacian and gradient and under certain conditions we show that harmonic quasiconformal maps are Lipschitz in the space and are bi-Lipschitz in the plane. Characterizations of harmonic quasiconformal maps by the boundary mappings are given in some settings.

# HYPERBOLIC DISTORTION FOR HOLOMORPHIC MAPS OF REGIONS

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Let $\Omega_j$ be a hyperbolic region with hyperbolic distance $h_j, j = 1, 2$. The talk treats two types of results. The first type deals with stronger versions of the general form of the Schwarz-Pick Lemma which asserts that a holomorphic map $f : \Omega_1 \to \Omega_2$ is a strict contraction relative to the hyperbolic distance unless it is a covering of $\Omega_1$ onto $\Omega_2$. It is known that the contracting property can be made quantitative in terms of sharp upper bounds involving the hyperbolic derivative at a point. New, sharp local lower bounds on the amount of contraction in terms of the hyperbolic derivative at a point are obtained. The second group of results deals with the special case in which $\Omega_1$ is simply connected region and concerns the hyperbolic distance between a holomorphic self-map and certain canonical self-maps of the region. For example, if $\Omega_1$ is a simply connected hyperbolic region and $f : \Omega_1 \to \Omega_2$ is a holomorphic function, let $\varphi_f$ be the unique holomorphic covering of $\Omega_1$ onto $\Omega_2$ with $\varphi_f(a) = f(a)$ and $\arg \varphi_f'(a) = \arg f'(a)$; if $f'(a) = 0$ take any value for $\arg \varphi_f'(a)$. Sharp upper and lower bounds on $h_2(f(z), \varphi_f(z))$ in terms of $h_1(a, z)$ are established. A related upper bound is given when the comparison function $\varphi_f$ is replaced by the conformal map $\psi_f$ of $\Omega$ onto a hyperbolic disk about $f(a)$ that satisfies $\psi_f'(a) = f'(a)$. The final comparison is with a two-sheeted branched covering.

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# ORLICZ-SOBOLEV SPACES WITH ZERO BOUNDARY VALUES ON METRIC MEASURE SPACES AND POINCARÉ INEQUALITIES

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In what follows, $X$ is a proper metric space equipped with a doubling Borel regular outer measure $\mu$, $\Omega \subset X$ is open and $\Psi$ is a Young function. For $E \subset X$ we denote by $N_0^{1,\Psi}(E)$ the space of all Orlicz-Sobolev functions with zero boundary values on $E$, introduced in [1] . Assuming that $\Psi$ is a doubling $N-$function, we obtain the following extensions from Newtonian spaces to Orlicz-Sobolev spaces of some density results from [2]:

1. Lipschitz functions with compact support in $\Omega$ are dense in $N_0^{1,\Psi}(\Omega)$ if locally Lipschitz functions are dense in $N^{1,\Psi}(X)$;

2. Every function in $N_0^{1,\Psi}(\Omega)$ is the limit in $N^{1,\Psi}(X)$ of a sequence of functions with bounded support contained in $\Omega$, provided that all functions in $N^{1,\Psi}(X)$ are $\Psi-$quasicontinuous.

The main result of this talk is the following $(\Psi,\Psi)-$Poincaré inequality for Orlicz-Sobolev functions with zero boundary values on balls, which generalizes well-known results involving $(p,p)-$Poincaré inequality.

**Theorem.** *Assume that $X$ supports a $\Phi-$Poincaré inequality in the sense of [3], for some strictly increasing Young function $\Phi$, and that $\Psi$ is a Young function such that $\Psi \circ \Phi^{-1}$ is a doubling Young function satisfying a $\nabla_2$-condition. Then there exists a constant $C > 0$ such that for every ball $B = B(x,r)$ with $r < \frac{diam(X)}{3}$, each $u \in N_0^{1,\Psi}(B)$ and every $\Psi-$weak upper gradient $g$ of $u$,*

$$\int_B \Psi\left(\frac{|u|}{r}\right)\, d\mu \le C \int_B \Psi(g) d\mu.$$

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# INJECTIVITY CONDITIONS IN THE COMPLEX PLANE

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A survey on some recent results concerning injectivity conditions in the complex plane,which are obtained by using certain geometric properties as starlikeness, spirallikeness, convexity, close-to-convexity. These results extend to continuously differentiable maps some well-known univalence conditions for analytic functions.

# ON ONE G. BOOLE'S IDENTITY FOR RATIONAL FUNCTIONS AND ITS APPLICATIONS

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In 1857 George Boole found an identity for a class of rational functions of a complex variable which, for a given function, connects the sum of its residues at finite points with the difference between the sums of its zeros and poles. In [1-2] a connection between the analog of this formula for Nevanlinna class functions and infinite-dimensional elliptic coordinates has been found. This coordinates in turn are closely connected with the spectra of finite-difference operators generated by infinite Jacoby matrices. Using these observations, we obtain some new trace formula for such operators. A more careful study of the analog of Boole's identity for Nevanlinna class functions leads to new results about the traces of differential operators (including the classical operators of Sturm-Liouville).

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# LOEWNER THEORY AND SCHWARZIANS IN $\mathbb{C}^n$

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The theory of Loewner chains and the Loewner differential equation were generalized to the several complex variables setting in 1974-75. The one-variable preSchwarzian, $f''(z)/f'(z)$, was generalized to higher dimensions and a specific type of chain was derived and used to give n-Dim preSchwarzian univalence and quasiconformal extension criteria. The problems of identifying what should play the role of Schwarzian derivative, $(f''(z)/f'(z))' - (1/2)(f''(z)/f'(z))^2$, and constructing $L$-chains to give $n$-dimensional generalizations of the Nehari, Ahlfors, Becker, Pommerenke, Epstein (and many others) Schwarzian derivative criteria proved to be a difficult challenge and remained unsolved.

We have solved this problem, Nov.08-present, with the construction of appropriate $L$-chains for the theory involving higher dimensional Schwarzian invariants. We now have n-Dim versions of the main Schwarzian univalence and qc-extension criteria of Nehari, Ahlfors, Becker, Epstein-Pommerenke and others (including pi-squared criteria).

The talk will focus on

(i) What are the Schwarzian invariants and how can they be derived?,

(ii) The many new Loewner chains constructed for deriving Schwarzian univalence criteria,

(iii) The role of affine/projective concepts and a Cartan matrix concept. Time constraints necessitate the emphasis being on describing the many results and motivation by analogies with the familiar classical one variable material.

# PROPERTIES OF THE PREIMAGES OF MONOTONE OPERATORS AND SOME LOCAL CONCEPTS WHICH ARE EQUIVALENT TO THEIR GLOBAL COUNTERPARTS

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We first observe that Minty-Browder monotonicity implies another type of monotonicity, as used by Church and Timourian, which we call *topological monotonicity.* This shows that locally injective Minty-Browder monotone operators are globally injective. We also prove that the local Minty-Browder monotonicity implies the global Minty-Browder monotonicity. This shows that the locally convex $C^1$ functions are actually globally convex.

We also enlarge the class of Minty-Browder monotone operators to the class of $h$-monotone operators and show that the preimages of such operators are indivisible by closed hypersurfaces. This property is good enough to ensure the global injectivity of $h$-monotone operators which are locally injective. Let us finally point out that local $h$-monotonicity does not imply the global $h$-monotonicity.

This talk is based on some joint works with G. Kassay, S. László and F. Szenkovits.

# BLOCH AND LANDAU'S THEOREMS ON HARMONIC AND BIHARMONIC MAPPINGS

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The aim of the lecture is to discuss the certain properties of planar harmonic mappings and in particular, we give a lower estimate for the Bloch constant for planar harmonic and biharmonic mappings. Also, for bounded planar harmonic and biharmonic mappings, we present improved versions of Landau's theorem. We also present similar results for $L(f)$, where $L$ represents the linear complex operator $L = z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}$ defined on the class of complex-valued $C^1$-functions in the plane and $f$ is an open harmonic mapping.

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# TENSOR PRODUCT OF ABSOLUTELY CONTINUOUS RESOLVENTS

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The (tensor) product of harmonic structure is done through the product of the associated semigroups. In order to obtain a standard $H$–cone we need an absolutely continuous resolvent. For a given semigroup $\mathcal{P}$, associated with an absolutely continuous resolvent, $\mathcal{P} \otimes \mathcal{Q}$ is associated with an absolutely continuous resolvent if and only if $\mathcal{P}$ is itself absolutely continuous.

A sufficient condition, weaker than previously known ones, is indicated on a (absolutely continuous) resolvent, such that the associated semigroup be also absolutely continuous.

# QUASISYMMETRIC DISTORTION SPECTRUM

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According to a well known result of Beurling and Ahlfors a quasisymmetric map of the line need not be absolutely continuous with respect to the Lebesgue measure. We study the singularity structure of such maps in terms of the quasiconformal extension. This amounts to finding bounds for the compression of Hausdorff dimension under quasisymmetric parametrization. A connection to multifractal properties of harmonic measure will shortly be discussed.

This talk is based on joint work with Stas Smirnov.

# BOUNDARY BEHAVIOR OF HARMONIC AND QUASIREGULAR MAPPINGS

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We study the connection between multiplicities of the zeros and boundary behavior of bounded harmonic and quasiregular mappings of the plane. For general introduction to harmonic mappings, see e.g. [1], [2] or survey articles [3, 4, 5].

Sufficient conditions for the existence of angular (non-tangential) limit at a boundary point, provided that multiplicities of zeroes of the function grow fast enough on a given sequence of points approaching the boundary, have been established in [6, 8]. We compare these results and also consider sharpness of such conditions in the planar case.

This talk is based on joint research with S. Ponnusamy [6, 7].


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# ON THE BELTRAMI EQUATIONS WITH 2 CHARACTERISTICS


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The existence problem for degenerate Beltrami equations is currently an active area of research. We prove a series of new criteria for the existence of homeomorphic $W^{1,1}_{loc}$ regular solutions to the Beltrami equations of the general form

$$\overline{\partial} f \;=\; \mu \partial f + \nu \overline{\partial f} \qquad (1)$$

and, in particular, to the Beltrami equations of the second type $\overline{\partial} f = \nu \overline{\partial f}$ which play a great role in many problems of mathematical physics.

Here we use the notations $\overline{\partial} f = (f_x + i f_y)/2$, $\partial f = (f_x - i f_y)/2$, $z = x + iy$, and $f_x$ and $f_y$ are partial derivatives of $f = u + iv$ in the variables $x$ and $y$, respectively. We call a homeomorphism $f \in W^{1,1}_{loc}(D)$ a **regular solution** of (1) if $f$ satisfies (1) a.e. and its Jacobian $J_f(z) \neq 0$ a.e.

**Theorem 1.** *Let $D$ be a domain in $\mathbb{C}$ and let $\mu$ and $\nu : D \to \mathbb{C}$ be measurable functions with $|\mu(z)| + |\nu(z)| < 1$ a.e. such that*

$$K_{\mu,\nu}(z) : = \frac{1 + |\mu(z)| + |\nu(z)|}{1 - |\mu(z)| - |\nu(z)|} \leq Q(z) \in FMO(D) . \tag{2}$$

*Then the Beltrami equation (1) has a regular solution.*

A function $\varphi : D \to \mathbb{R}$ is said to be of **finite mean oscillation** in $D$, $\varphi \in$ **FMO(D)**, if the mean deviation of $\varphi(z)$ from its mean value over the infinitesimal disk $B(z_0, \varepsilon)$ is finite at every point $z_0 \in D$.

**Corollary 1.** *In particular, (1) has a regular solution if*

$$\overline{\lim_{\varepsilon \to 0}} \frac{1}{\pi \varepsilon^2} \int\limits_{B(z_0,\varepsilon)} K_{\mu,\nu}(z)\, dxdy < \infty \qquad \forall\, z_0 \in D . \tag{3}$$

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# UNIFORM CONTINUITY AND $\varphi$-UNIFORM DOMAINS

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This talk is based on the manuscript [1]. Let us consider the Lipschitzian modulus of continuity of the identity mapping

$$id : (G, d_1) \to (G, d_2), \tag{1}$$

where $(G, d_i)$, $i = 1, 2$, are metric spaces in $\mathbb{R}^n$. We obtain a sharp bound for the Lipschitzian modulus of continuity of the identity mapping (1) when $d_2$ is the Euclidean and $d_1$ the quasihyperbolic metric. A similar result also holds when $d_1$ is the distance ratio metric.

In 1979, the notion of uniform domains were introduced by Martio and Sarvas [2]. In 1985, a generalization of uniform domains were introduced in [3] so-called the $\varphi$-uniform domains. We obtain necessary and sufficient conditions for $\varphi$-uniform domains $G \subsetneq \mathbb{R}^n$ in terms of Lipschitzian modulus of continuity of the identity mapping (1) when $d_2$ is the quasihyperbolic metric and $d_1$ is the distance ratio metric.

In addition, we discuss certain properties of $\varphi$-uniform domains; in particular, we prove that $\varphi$-uniform domains are preserved under bilipschitz mappings of $\mathbb{R}^n$.

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# ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS AND DIFFERENTIAL OPERATORS

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We say that a function of the form $f(z) = z - a_2z^2 - a_3z^3 - ..., a_j > 0$ that is analytic in the unit complex disc $U = \{z; |z| < 1\}$, is a function with negative coefficients. We consider Ruscheweyh's operator $\mathcal{D}^n$, where

$$\mathcal{D}^n f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!}, \quad n \in \mathbb{N}$$

and the operator $D^n$ defined by a) $D^0 f(z) = f(z)$; b) $D^1 f(z) = Df(z) = zf'(z)$; c) $D^n f(z) = D(D^{n-1}f(z))$, $n \in \mathbb{N}$ (see [1] and [2]).

Using this differential operators we define and study some classes of functions with negative coefficients.

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# FIBER STRUCTURE OF THE TEICHMÜLLER SPACE OF A BORDERED RIEMANN SURFACE

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Let $\Sigma^B$ be a Riemann surface whose boundary consists of n curves homeomorphic to the circle. By sewing on $n$ copies of the unit disk one obtains a compact Riemann surface $\Sigma^P$. We show that the Teichmüller space of $\Sigma^B$ is a complex fiber space over the Teichmüller space of $\Sigma^P$. The fibers have a simple function-theoretic description in terms of a class of non-overlapping conformal maps with quasiconformal extensions. As an application, we obtain new local coordinates for the Teichmüller space of bordered Riemann surfaces.

The formulation and proofs of these results are motivated by two-dimensional conformal field theory.

# ANOTHER LOOK AT THE SCHWARZ LEMMA

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There is a long history associate with the rigidity problems for holomorphic mappings at their interior or boundary fixed points, the work of Schwarz, Pick, Julia and Wolff being among the most important.

In this talk we present an exposition of old and new results related to the classical Schwarz Lemma and its different modifications especially in the spirit of the structure of the fixed point set. In particular, we discuss some new aspects of the asymptotic behavior of one-parameter semigroups of holomorphic mappings and present angular characteristics of their trajectories at their Denjoy–Wolff points, as well as at their regular repelling points (whenever they exist). This enables us to establish new rigidity properties of holomorphic generators via the asymptotic behavior of the semigroups they generate.

# INVARIANT SCHWARZIAN DERIVATIVE OF HIGHER ORDER AND ITS APPLICATIONS

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For a nonconstant holomorphic map between projective Riemann surfaces with conformal metrics, we can consider the classical Schwarzian derivative and the invariant Schwarzian derivative. We show that these two quantities are related in terms of the "Schwarzian derivative" of the metrics. We can extend this scheme to higher-order analogues of the Schwarzian derivative and obtain new invariants for the metric and the holomorphic map. When the source surface is the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with hyperbolic metric, the invariant for a function $f : \mathbb{D} \to \widehat{\mathbb{C}}$ is given as

$$Vf(z) = (Sf)'(z) - \frac{4\bar{z}}{1-|z|^2} Sf(z),$$

where $Sf$ is the Schwarzian derivative $(f''/f')' - (f''/f')^2/2$ of $f$. The norm of $Vf$ is defined by

$$\|Vf\|_3 = \sup_{z\in\mathbb{D}} (1-|z|^2)^3 |Vf(z)|.$$

Then we have an analog to Nehari's univalence criteria in the following form.

**Theorem** *Let $f$ be a nonconstant meromorphic function on the unit disk $\mathbb{D}$. If $f$ is univalent in $\mathbb{D}$, then $\|Vf\|_3 \le 16$. The number 16 is sharp. Conversely, if $\|Vf\|_3 \le 3/2$, then $f$ is univalent in $\mathbb{D}$.*

Unfortunately, the latter part is weaker than the Nehari univalence criterion in terms of the Schwarzian derivative. We hope that new methods will be invented to improve the above univalence criterion.

# EXTENDABILITY OF CLASSES OF MAPS AND NEW PROPERTIES OF UPPER SETS

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Upper sets were defined in [1, 2] where we studied extendability properties of subsets of $R^n$ under bilipschitz, quasisymmetric and quasiconformal maps with bilipschitz coefficient or coefficient of quasiconformality near to 1.

Assume that $A \subset R^n$. The upper space, or a space of balls $A \times R_+$ is supplied with a metric $\varrho$, invarian under similarities of $R^n$. We define the upper set $\widetilde{A} = \{(x,r) \in A \times R_+ : \exists y \in A \setminus \{x\}, r = |x-y|\}$. A sequence $\bar{u} = \{u_0, \ldots, u_N\}$ in $A \subset R^n$ is a $\lambda$-sequence iff $\varrho(u_{i-1}, u_i) \leq \lambda$, $i = 1, \ldots, N$. It connects $u_0$ and $u_N$. $\widetilde{A}$ is $\lambda$-connected if any pair of points is connected by a $\lambda$-sequence in $\widetilde{A}$. If $\widetilde{A}$ is not $\lambda$-connected then the set of $\lambda$-components is well-structured: it was proven in [1] that it forms a family tree graph. This property is important in questions of extendability.

Here we continue the study of the structure of upper sets. The $\lambda$-metric is defined on $\lambda$-components of $\widetilde{A}$ as follows. The length of a $\lambda$-sequence $\bar{u}$ is $\sum_{i=1}^{N} \varrho(u_{i-1}, u_i)$. The $\lambda$-distance $\varrho_\lambda(u,v)$ is the infimum of lengths of $\lambda$-sequences connecting $u$ and $v$. A $\lambda$-connected set is $(K,\lambda)$-quasiconvex if $\varrho_\lambda(u,v) \leq K\varrho(u,v)$. We generalize homotopy notions to discrete space and introduce $\lambda$-homotopy and $\lambda$-fundamental groups. A $\lambda$-connected space is $\lambda$-simply connected if its $\lambda$-fundamental group is trivial.

**Theorem 1.** *Assume that $\lambda \geq 1$. Then every $\lambda$-component of the upper set $\widetilde{A}$ is $(2, \lambda+1)$-quasiconvex.*

**Theorem 2.** *Assume that $\lambda \geq 1$. Then every $\lambda$-component of the upper set $\widetilde{A}$ is $(\lambda+1)$-simply connected.*

We need these properties to extend certain characteristics of $\widetilde{A}$ onto the space of balls $R^n \times R_+$ and to extend any map $f : A \to R^n$ of the above classes onto a map $F : R^n \to R^n$.

Acknowledgements. The authors were supported by the Russian foundation for Basic Research.

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# ON GENERALIZED GAUSS AND BOCHNER-MARTINELLI MEANS

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In this work integral means and properties of singular kernels are studied. By generalizing the Gauss mean-value, respectively, the Bochner-Martinelli, formula, pseudospherical, respectively, pseudoradial, functions, are introduced. Characterizations of such functions, conditions for holomorphicity, and constancy criterion for semiharmonic functions are obtained. In particular, it is shown that *if $D$ is a relatively compact domain in a normal Riemann domain, any two locally Lipschitz* *functions on* $\overline{D}$ *with (local) almost everywhere equal a-radial and a-tangential derivatives at almost every* $a \in D$ *differ at most by a constant.*

ACKNOWLEDGEMENTS. The author was supported by the "Globale Methoden in der komplexen Geometrie" Grant of the German research society DFG and a Faculty Improvement Grant of Minnesota State University, Mankato.

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# THE HAWAII CONJECTURE AND RELATED PROBLEMS

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The object of the talk is to present a proof of the following statement which was conjectured by T. Craven, G. Csordas and W. Smith [1] and is known now as the Hawaii conjecture.

**Hawaii conjecture.** *If a real polynomial* $p$ *has exactly* $2m$ *nonreal zeros, counting multiplicities, then its logarithmic derivative has at most* $2m$ *critical points, counting multiplicities.*

We show that this conjecture is true not only for real polynomials but also for all real entire functions of genus $1^*$ with finitely many nonreal zeros. A real entire function $f$ is of genus $1^*$ if $f(z) = e^{-az^2}g(z)$, where $a \geqslant 0$ and $g$ is a real polynomial or a real entire function of genus 0 or 1.

In the talk, we will also discuss some open problems concerning generalizations of the Hawaii conjecture and the Newton inequalities.

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## ON GENERALIZED LAPLACE EQUATION AND NONLINEAR OPERATORS

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This work deals with the nonlinear potential theory, particularly with the techniques of the construction of nonlinear resolvent associated with a given nonlinear operator. After some introductory remarks about the Dirichlet problem for the generalized Laplace equation we define a nonlinear operator on the space of the essentially bounded functions on an open bounded subset of $k$ dimensional real space and we associate with it a sub-Markovian nonlinear resolvent.

## REGION OF VARIABILITY FOR EXPONENTIALLY CONVEX UNIVALENT FUNCTIONS

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For $\alpha \in \mathbb{C} \setminus \{0\}$ let $\mathcal{F}_\alpha$ denote the class of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ with $f(0) = 0 = f'(0) - 1$ and

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)} + \alpha z f'(z)\right) > 0 \quad \text{in } \mathbb{D}.$$

For any fixed $z_0$ in the unit disk and $\lambda \in \overline{\mathbb{D}}$, we shall determine the region of variability $V(z_0, \lambda)$ for $\log f'(z_0) + \alpha f(z_0)$ when $f$ ranges over the class

$$\mathcal{F}_\alpha(\lambda) \quad = \quad \left\{ f \in \mathcal{F}_\alpha \colon f''(0) = 2\lambda - \alpha \right\}.$$

We geometrically illustrate the region of variability $V(z_0, \lambda)$ for several sets of parameters.

This is a joint work with Prof. S. Ponnusamy and Prof. M. Vuorinen.

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# REPRESENTATION FOR THE SOLUTION OF THE RIEMANN — HILBERT PROBLEM AS SCHWARZ — CHRISTOFFEL INTEGRAL

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The Riemann — Hilbert problem for function $\mathcal{P}^+(z)$, $z = x + iy$, analytic in the upper half–plane $\mathbb{H}^+$, is considered,

$$\operatorname{Re}\left[\, h(x)\,\mathcal{P}^+(x)\,\right] = c(x), \qquad x \in \mathbb{R}, \tag{1}$$

where $h(x)$, $c\,(x)$ are piece–wise Hölder coefficients with dicontinuities in points $x_0 = \infty$, $x_1, \ldots,$ $x_K$ on the real axis $\mathbb{R}$. Let intoduce notations for jumps of argument of coefficient $h$: $\delta_0 = [\arg h(+\infty) - \arg h(-\infty)]/\pi$, $\delta_k = [\arg h(x_k + 0) - \arg h(x_k - 0)]/\pi$, $k > 0$, and notations for their integral and fractional parts: $\mu_k = [\delta_k]$, $\alpha_k = \{\delta_k\}$. Denote $\widetilde{\mathbb{H}} = \overline{\mathbb{H}^+} \setminus \{x_k\}$. The solution $\mathcal{P}^+ \in C(\widetilde{\mathbb{H}})$ is subjected to the conditions: $\mathcal{P}^+(z) = = \mathcal{O}(z^{\alpha_0 + n_0})$, $z \to x_0$; $\mathcal{P}^+(z) = \mathcal{O}\big[(z - x_k)^{\alpha_k - n_k}\big]$, $z \to x_k$, if $n_k \neq 0$, and $\mathcal{P}^+(z) = \mathcal{O}(1)$, $z \to x_k$, if $n_k = 0$. Here $n_0, n_1, \ldots, n_K$ are prescribed nonnegative integers.

**Theorem 1.** *1) If the index $\mu := n_0 - \mu_0 + \sum_{k=1}^{K}(\mu_k + n_k)$ is nonnegative, than solution $\mathcal{P}^+$ of problem (1) has the form*

$$\mathcal{P}^+(z) = \mathcal{X}^+(z)\Big[P_\mu(z) + \mathcal{F}^+(z)\Big], \tag{2}$$

*where $P_\mu(z)$ is an arbitrary polynomial of power $\mu$ with real coefficients, $\mathcal{X}^+(z) := ie^{-\arg h_K} \prod_{k=1}^{K}(z - x_k)^{\alpha_k - n_k}$ is a canonical function, and $\mathcal{F}^+$ is defined by the formula*

$$\mathcal{F}^+(z) := \sum_{k=0}^{K} \frac{c_k(z-\lambda_k)^\mu}{h_k \pi i} \int\limits_{\mathcal{L}_k} \frac{(t-\lambda_k)^{-\mu}}{\mathcal{X}^+(t)(t-z)}\, dt; \tag{3}$$

*here $\mathcal{L}_0 := (-\infty, x_1)$, $\mathcal{L}_k := (x_k, x_{k+1})$, $\mathcal{L}_K := (x_K, +\infty)$, $\lambda_k \in \mathbb{R} \setminus \mathcal{L}_k$.*

*2) If $\mu = -1$, than the unique solution of (1) is given by the formula (2) with $P_\mu \equiv 0$, and with $\mu = 0$ in (3). If $\mu < -1$, than for solvability of (1) the following conditions are necessary and sufficient: $\int\limits_{\mathbb{R}} \frac{t^n c(t)}{h(t)\, \mathcal{X}^+(t)}\, dt = 0$, $n = \overline{0, |\mu| - 2}$. The solution of (1) is given by the same formula as for $\mu = -1$.*

The main result of this talk is the following theorem.

**Theorem 2.** *If coefficients $h(x)$ and $c(x)$ are piece–wise constant, than the representation (2) for $\mathcal{P}(z)$ can be transformed to the form of generalized Schwarz — Christoffel integral*

$$\mathcal{P}^+(z) = ie^{-\arg h_K} \int_{z_*}^{z} \prod\nolimits_{k=1}^{K} (t - x_k)^{\alpha_k - n_k - 1} R(t)\, dt + w_*, \tag{4}$$

*where $R$ is a polynomial with real coefficients.*

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## BOUNDARY INTEGRAL EQUATIONS FOR TWO–DIMENSIONAL LOW REYNOLDS NUMBER FLOW PAST A POROUS BODY

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In this lecture, we use the method of matched asymptotic expansions for the two-dimensional steady flow of a viscous incompressible fluid at low Reynolds number past a porous body of arbitrary shape. We assume that the flow inside the porous body is described by the Brinkman model, i.e., by the continuity and Brinkman equations, and that the velocity and boundary traction fields are continuous across the interface between the fluid and porous media. By employing some indirect boundary integral representations, the inner problems are reduced to uniquely solvable systems of Fredholm integral equations of the second kind in some Sobolev or Hölder spaces, while the outer problems are solved by using the singularity method. It is shown that the force exerted by the exterior flow on the porous body admits an asymptotic expansion with respect to low Reynolds number, whose terms depend on the solutions of the above mentioned system of boundary integral equations.

By using the Oseen flow in the exterior domain, it can be shown that the Stokes–Brinkman expansion converges in any compact region to the Oseen–Brinkman solution if the Reynolds number tends to zero, in a similar manner as shown by G.C. Hsiao and R.C. MacCamy in 1973 and 1982 for flows around rigid obstacles.

1. M. Kohr, W.L. Wendland, G.P. Raja Sekhar, "Boundary integral equations for two-dimensional low Reynolds number flow past a porous body", Mathematical Methods in the Applied Sciences, **32**, 922-962 (2009).

# A DISTORTION THEOREM FOR FUNCTION CONVEX IN ONE DIRECTION

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We start with a short overview of some class of univalent functions that recently found applications in complex dynamics. These applications are based, inter alia, on distortion theorems.

We present a new distortion result for functions convex in one direction. This result comprises, in fact, a criterion for a function convex in the positive direction of real axis to have bounded imaginary part.

The talk is based on a joint work with M. Elin and D. Shoikhet.

# INVERSE SPECTRAL PROBLEMS FOR SINGULAR RANK-ONE PERTURBATIONS OF A HILL OPERATOR

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We investigate an inverse spectral problem for the singular rank-one perturbations of a Hill operator. We give a necessary and sufficient condition for a real sequence to be the spectrum of a singular rank-one perturbation of the Hill operator.

# TEICHMÜLLER SPACE OF AN ORIENTED JORDAN CURVE IN THE EXTENDED COMPLEX PLANE

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One of the most powerful tools, when studying Riemann surface, is the notation of Teichmüller space, i.e. a metrizable and complete quotient space of closed Riemann surfaces with genus $g > 2$. While the concept was introduced by ingenious German mathematician O. Teichmüller before World War II, the name appears because of L. Bears and L. V. Ahlfors in the late fifties. The function-theoretic model of this, not easy understandable, original Teichmüller space, was built up by the use of equivalence classes of quasiconformal automorphisms of the unit disc or its boundary representation called quasihomographies. Making use of the Poisson integral extension operator one may construct harmonic representation of the universal Teichmüller space in which, particular, boundary normalized harmonic automorphisms of the unit disc, represent elements of the space, in question.

The main purpose of the lecture is to present a number of theorems and constructions regarding metric and topological feature of harmonic and quasihomographic models of the universal Teichmüller space. It is word to mention that the harmonic representation of the universal Teichmüller space was designed by E. Paprocki – a young mathematician killed in a road accident before his, highly estimated, doctorate. His idea links once again extremal quasiconformal automorphisms of the unit disc with two classes of analytic functions, defined in the unit disc called the conjugate Paprocki spaces of analytic functions. Some basic properties of functions from those spaces will also be presented during this lecture.

# CONTINUOUS MAPPINGS OF DOMAINS ON MANIFOLDS

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**Theorem 1** Let $f : \bar{D} \to D_1$ be continuous mapping of domains ($D \subset M^n$, $D_1 \subset N^n$, $\bar{D} \neq M^n$ – be domains on $n$-dimensional manifolds) with following properties:

1) for some open portion $U$ of boundary $\partial D$ restriction $f_U : U \to f(U)$ be $k$-to-one covering mappings;

2) $f(\partial D) \cap f(D) = \emptyset$;

3) $f(U) \cap f(\partial D \setminus U) = \emptyset$.

Then $f|_D$ be interior mapping or there exists point $y \in f(D)$ having at least $|k| + 2$ preimage.

If $f$ be zerodimensional mapping then a set of points in second case having at least $|k| + 2$ preimage have dimension $n$.

**Theorem 2** Let $M^2$ be closed twodimensional manifold. There exists continuous mappings $f : M^2 \to B^2$ in twodimensional ball that for every point $y \in B$ card $f^{-1}y \leq 2$.

**Theorem 3** Let $RP^n$ be $n$-dimensional projection space. There exists continuous mappings $f : RP^n \to B^n$ in $n$-dimensional ball that card $f^{-1}y \leq 2^{n-1}$ for every point $y \in B^n$.

**Open problem** Do there exist an continuous mapping of $n$-dimensional projective space $RP^n$ on $n$-dimensinal sphere $S^n$ that for every point $y$ of the sphere preimage $f^{-1}y$ contains no more the three points if $n > 2k$?

1. A. K. Bakhtin, G. P. Bakhtina, Yu.B. Zelinskii, "Topological - algebraic structure and geometrical methods in complex analysis," Kyiv, Inst. of math. NANU. - 2008. - 308 p. (in Russian)

# INTERIOR CAPACITIES OF CONDENSERS WITH INFINITELY MANY PLATES IN LOCALLY COMPACT SPACES

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The study deals with the theory of interior capacities of condensers in a locally compact Hausdorff space, a condenser being treated here as a countable, locally finite collection of arbitrary sets with the sign $+1$ or $-1$ prescribed such that the closures of opposite-signed sets are mutually disjoint. We are motivated by the fact, discovered recently, that in the noncompact case the main minimum-problem of the theory is in general unsolvable and this occurs even under very natural assumptions (e. g., for the Newtonian, Green, or Riesz kernels in $\mathbb{R}^n$, $n \geqslant 2$, and closed condensers of finitely many plates); compare with [1]. Necessary and sufficient conditions for the problem to be solvable were given in [2, 3].

Therefore it was particularly interesting to find statements of variational problems *dual* to the main minimum-problem of the theory of interior capacities of condensers (and hence providing some new *equivalent* definitions of the capacity), but now *always solvable* (e. g., even for nonclosed, unbounded condensers of infinitely many plates).

For all positive definite kernels satisfying B. Fuglede's condition of consistency between the strong and weak* topologies, problems with the desired properties have been posed and solved (see [4-6]).

Their solutions provide a natural generalization of the well-known notion of interior equilibrium measure associated with a set. We give a description of those solutions, establish statements on their uniqueness and continuity, and point out their characteristic properties (see [5, 6]).


1. B. Fuglede, "On the theory of potentials in locally compact spaces," Acta Math., **103**, No. 3-4, 139-215 (1960).
2. N. Zorii, "On the solvability of the Gauss variational problem," Comput. Meth. Funct. Theory, **2**, No. 2, 427-448 (2002).
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5. N. Zorii, "Interior capacities of condensers in locally compact spaces," arXiv:0902.0396, 1-38 (2009); submitted to Potential Analysis.
6. N. Zorii, "Interior capacities of condensers with infinitely many plates in a locally compact space," arXiv:0906.4522, 1-42 (2009).

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