Remarks on the global synchronisation problem

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Synchronisation as a natural phenomenon

Self-synchronisation observed in various systems with numerous agents

Christaan Huygens
(1629-1695)

examples :

- fireflies,
- oscillators,
- clapping, etc.

How can a collection of interacting particles can oscillate in pace if no external stimulation is received ?

(A)synchronous cellular automata

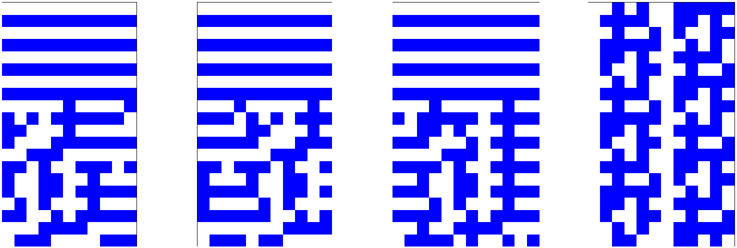
How the different levels of a model can interact ?
Starting from a disordered state, we want a macroscopic behaviour in the form of a regular oscillation

Introduced by Das, Crutchfield, Mitchell & Hanson:
`Evolving globally synchronized cellular automata`, 1995.

There exists an abundant litterature with continous models... but
$\rightsquigarrow$ what about discrete cellular systems?

Global synchronisation problem

one-dimensional binary systems with periodic boundary conditions.
goal: for **any initial condition**, the system eventually oscillates in the cycle with two homogeneous configurations (**0** and **1**).



Example of rule 1078270911 with radius 2.

A few properties

- A solution is not color-blind.
- A trajectory can not contain two configurations that are in the same class of rotations.
- The height of a solution of size n is bounded by $\chi(n) - 2$, where χ is the number of classes of rotations.

n	1	2	3	4	5	6	7	8	9
$\lvert\chi(n)\rvert$	2	3	4	6	8	14	20	36	60

First step : ECA space

- There exists no ECA which achieves synchronisation.

Proof:

000	001	100	101	010	011	110	111
a	b	c	d	e	f	g	h

a=1 ; h=0 ; d = e = 0 ; etc.

$\rightsquigarrow$ but... difficult to generalise, even to a neighbourhood of size 4

Formulation as a SAT problem

How to translate our CA problem as a satisfiability problem ?
SAT : `is there an assignment of boolean variables that makes a given formula true ?`
Sat solvers : accept entries in conjunctive normal form (CNF)

plan : for a given neighb. size, ask for more and more initial conditions to be synchronised until

- either we know that there are no solutions for this neighb.,
- or we find a good candidate to solve the problem for every size

$\rightsquigarrow$ How to "code" the synchronisation of a set of configurations ?

SAT coding (example for the ECA case)

Two types of boolean variables :

the transition rule :

000	001	100	101	010	011	110	111
t_0	t_1	t_2	t_3	t_4	t_5	t_6	t_7

evolution of initial condition:

```
abcdefgh
ijklmnop
qrstuvwx
```

- Blinking condition :

$$F_{\mathrm{b}} = t_0 \wedge \bar{t_7}.$$

- Coding of the initial condition $x = 00101011 \leftrightarrow \texttt{abcdefgh}$:

$$F_{\mathrm{ic}}(x) = \bar{a} \wedge \bar{b} \wedge c \cdots \wedge h$$

- Synchronisation condition:

$$F_{\mathrm{synch}}(x) = (q = r) \wedge (r = s) \wedge \cdots \wedge (w = x)$$

Consistency conditions

how do we code the coherence of evolution and transition table ?

```
00101011 abcdefgh
00010111 ijklmnop
```

For $t = 0, i = 4$, write : $F_{\text{cons}}(x, t, i) = c \wedge \bar{d} \wedge e \wedge (l = t_5)$
transformed into: $F_{\text{cons}}(x, t, i) = c \wedge \bar{d} \wedge e \wedge (l \vee \bar{t_5}) \wedge (\bar{l} \vee t_5)$

but next state not known, there are 8 possibilities for each local transition...

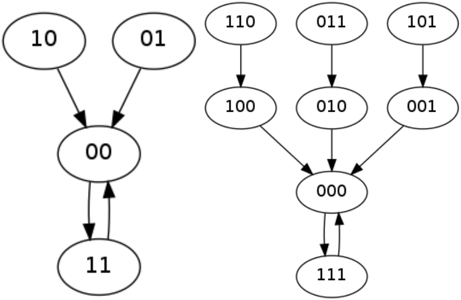
see the proceedings for technical details

Results

use of the `minisat` solver: `http://minisat.se/`
ECA case The maximum synch. length is 4
For $\mathcal{S} = \{2, 3\}$, we find that:

- $(1, 127)$ have height of 1,
- $(19, 55)$ and $(9, 65, 111, 125)$ have height of 2, and

rule 9:

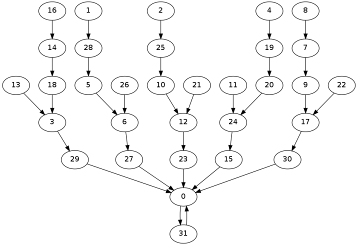


$k = 4$ case

The maximum synchronisation size is 7.
For $\mathcal{S} = \{2, \ldots, 6\}$, we find that:

- 3 rules of height 4:
  (1077,21471),
  (4427,11639),
  (11893,20875),
- 2 rules of height 5:
  (1205,17461,21215,21469),
  (5419,11095),
- 1 rule of height 6:
  (4363,12151).

rule 5419 :



$k = 5$ case (radius 2)

Size of the space $\#(2,5) = 2^{2^5} = 2^{32} \approx 4.10^9$.
direct exhaustive exploration becomes tricky...
SAT probelm : 74768 variables and 4563060 clauses... (long!)

with minisat : formula our best result: rule 1078270911, synch.
size interval $\mathcal{S} = \{2, \ldots, 11\}$
Rule has a height of 18.
new !: For $\tau = 20$, **only rule** which synch. sizes 2 to 11.
Test fails for size 12.

$\rightsquigarrow$ conjecture : max. synch. length is 11

The stochastic case

What happens if the transitions are not deterministic ?

000	001	100	101	010	011	110	111
a	b	c	d	e	f	g	h

Now take $a, \ldots, h$ as the probability to update to 1.
we still have $a = 0$ and $h = 0$...

Almost all rules are solutions!

000	001	100	101	010	011	110	111
a	b	c	d	e	f	g	h

- Any rule such that $b, \ldots, g \in (0,1)$ is a solution.

Proof:
$\mathbf{0} - \mathbf{1}$ is the only set of attractive configurations
non-zero probability to reach $\mathbf{0} - \mathbf{1}$ in less than $n/2$ steps.
idea : evolution may follows exactly (Z_t) and reach $\mathbf{0}$ or $\mathbf{1}$.

(Z_t) is the process such that:

$$Z_t(0,0,0) = 1,\ Z_t(1,1,1) = 0,$$
$$Z_t(a,b,c) = \begin{cases} 0 & \text{if } t \text{ is even} \\ 1 & \text{if } t \text{ is odd} \end{cases}$$

example:
00001111000
11100000011
00111111110
10000000000
11111111111

"Efficient" solutions

Measuring efficiency: T(x): synch. time ; random var.
Expected average synch. time : $EAST(n) = 1/2^n \sum_{x \in E_n} T(x)$
Worst expected synch. time : $WEST(n) = \max_{x \in E_n}$

- There are rules for which $WEST(n) = \mathcal{O}(n^2)$

Proof:
consider local rule : take the α-asynchronous shift and invert it.
If we apply an "inversion mask" at each time step, the dynamics is the same as the α-asynchronous shift

$\rightsquigarrow$ Does there exist (qualitatively) more efficient rules?

To sum up

Although easy to solve "statistically", the global synchronisation problem is interesting when we search for perfect solutions.

Formulation of the problem as a SAT problem allows us to harvest some results for $k = 3$, $k = 4$ and $k = 5$.

No obvious property emerges for the visual inspection of the solutions.

- There exists a huge gap with stochastic CA : as soon as one is allowed to use noise, it becomes "easy" to solve the problem !

Questions

Does there exist a solution to the global synchronisation problem?
For a given neighbourhood, what can we say about the maximum synchronisation size ?

How can we go further with the SAT formulation?
change the coding of the problem ?
e.g. define synch with trees rooted on $\mathbf{0}$ and $\mathbf{1}$
Can we use the SAT approach for exploring other questions ?
e.g. nilpotency

Are there stochastic solutions with a linear convergence time ?