Universal Time-symmetric Number-Conserving Cellular Automaton

Diego Maldonado[1] Andrés Moreira[2] Anahí Gajardo[1]

[1]Universidad de Concepción, Chile
[2]Universidad Técnica Federico Santa María, Chile

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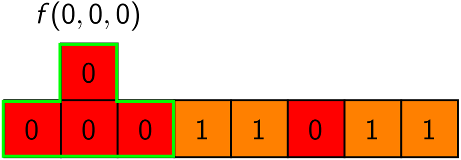
Definition

A **cellular automaton** (CA) is a function $F \in \mathrm{End}(Q^{\mathbb{Z}})$ defined by (N, f), where $N \subset \mathbb{Z}$ finite is called neighborhood, and $f : Q^N \to Q$ is such that $F(c)_i = f(c_{N+i})$.



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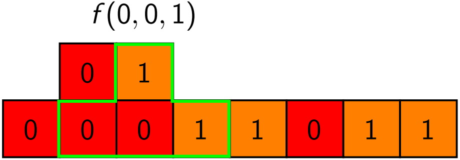

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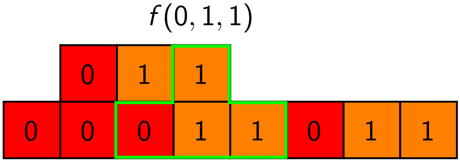

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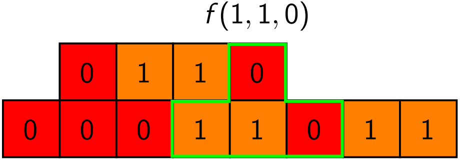

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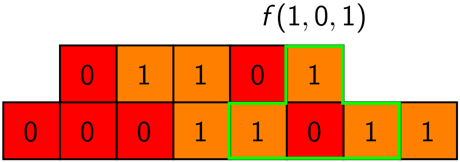

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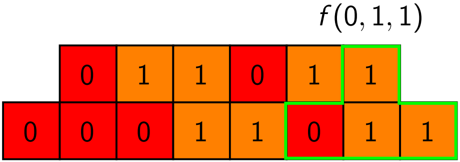

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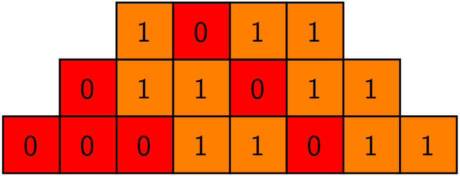

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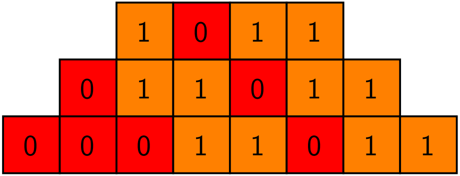

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Number Conservation

$F \in \text{End}(Q^{\mathbb{Z}})$, with $Q = \{0, ..., s-1\}$ is **Number-Conserving** (NCCA) if

$$\forall \alpha \in Q^{\mathbb{Z}} \text{ finite}, \sum_{i \in \mathbb{Z}} F(\alpha)_i = \sum_{i \in \mathbb{Z}} \alpha_i$$

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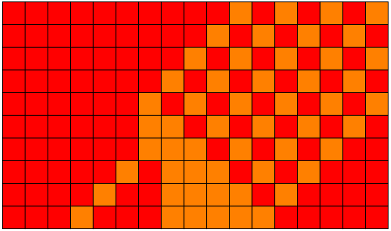

Example: Vehicular traffic rule.

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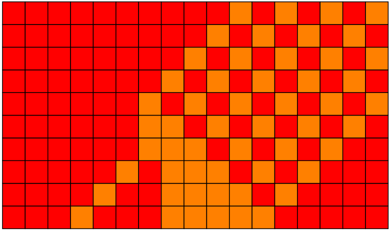

Example: Vehicular traffic rule.

The class of reversible NCCA is closed under $\circ$ and $()^{-1}$.

How restricted are reversible NCCA?

- All reversible one-way 2 states NCCA are trivial[1].

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How restricted are reversible NCCA?

- All reversible one-way 2 states NCCA are trivial[1].
- There is a non-trivial reversible NCCA with neighborhood of size 3[2].
- There is a reversible NCCA with neighborhood of size 4 that simulate any Turing machine[3].

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Time Symmetry

We say that a RCA F is **Time-Symmetric** (TSCA) if there exists a reversible CA H such that

$$F \circ H = H \circ F^{-1} \text{ and } H \circ H = Id \text{ i.e. } H \text{ is an involution}$$

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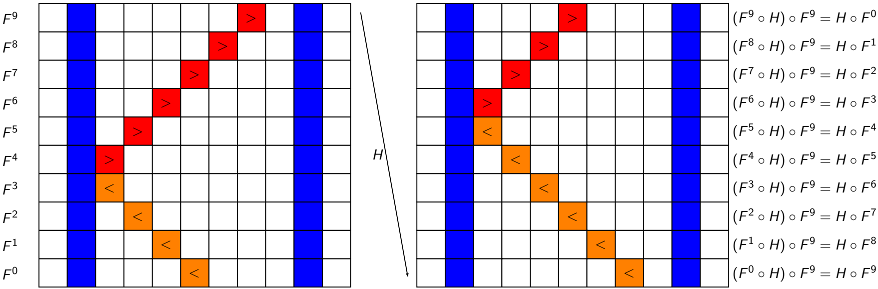

Example: Time-Symmetric CA

There exists TSCA?

- There is a non-trivial TSCA with neighborhood of size 3 and 2 states[4].

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- There is an intrinsically universal TSCA with neighborhood of size 3[5].

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There exists TSCA?

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- There is an intrinsically universal TSCA with neighborhood of size 3[5].
- Are there any TSNCCA? How complex are they?

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First approach: simulating an RCA by a TSCA

Theorem (Gajardo, Kari and Moreira (2012))

Given F a RCA with states Q, $F \times F^{-1}$ with states Q^2 is a TSCA and simulates F

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Proof (Idea).

$$\begin{array}{ccc}
(\alpha, \beta) & \xrightarrow{(F \times F^{-1})^{-1} = F^{-1} \times F} & (F^{-1}(\alpha), F(\beta)) \\
H \downarrow & & \downarrow H \\
(\beta, \alpha) & \xrightarrow{F \times F^{-1}} & (F(\beta), F^{-1}(\alpha))
\end{array}$$

□

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□

Problem

Even if $Q \subset \mathbb{Z}$, we cannot talk about Number conservation in Q^2.

Morita's Work

Theorem (Morita (2012))

Given a reversible CA F

$$\Downarrow$$

There exists a RNCCA $\overline{F}$ that simulates F

Morita's Work

Theorem (Morita (2012))

Given a reversible CA F (partitioned)(one way).

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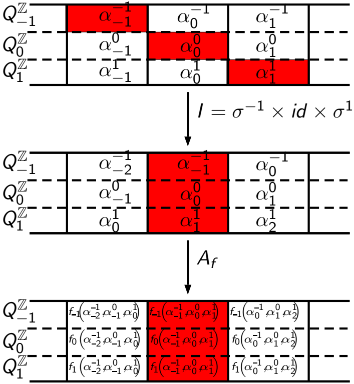
There exists a RNCCA $\overline{F}$ that simulates F (neighborhood of size 4).

Partitioned CA

$F \in \mathrm{End}(Q^{\mathbb{Z}})$ with $Q = Q_{n_1} \times \ldots \times Q_{n_m}$ is a **partitioned CA** if it is a composition of an autarkic CA A_f with local function f and $I := \sigma^{n_1} \times \ldots \times \sigma^{n_m}$, i.e., $F = A_f \circ I$



Proposition

A partitioned CA is reversible (RPCA) if and only if f is bijective.

Morita's Work

Theorem

Given a reversible CA F (partitioned)(one way)

$$\Downarrow$$

There exists $\overline{F}$ a RNCCA that simulates F (neighborhood of size 4).

Morita's Work

Theorem

$$\textit{Given a reversible CA } F \underbrace{\textit{(partitioned)(one way)}}_{\textit{No TS}}$$

$$\Downarrow$$

There exists $\overline{F}$ *a RNCCA that simulates* F *(neighborhood of size* 4*).*

M., Gajardo and Moreira

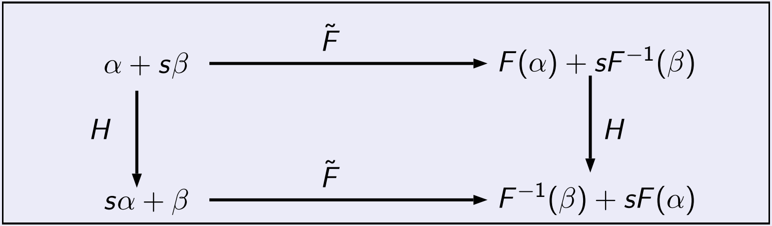
Theorem

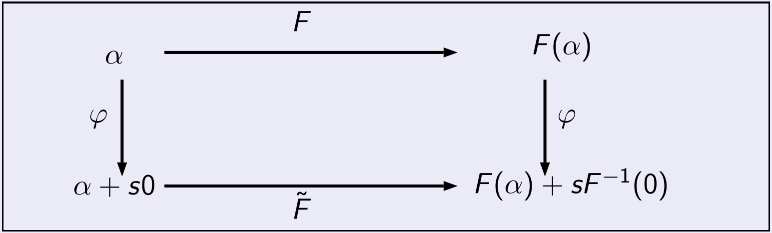
Given a RNCCA $F \in End(\{0, ..., s-1\}^{\mathbb{Z}})$ *with neighborhood of size n*

$$\Downarrow$$

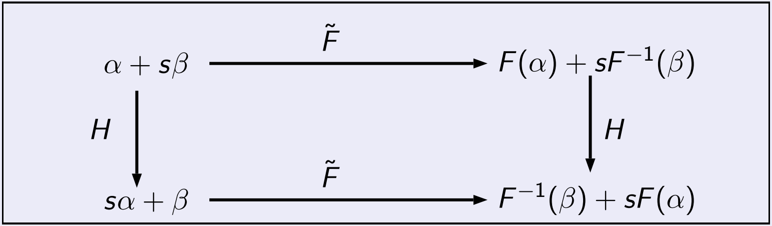
There exists $\tilde{F} \in End(\{0, ..., s^2-1\}^{\mathbb{Z}})$ *a TSNCCA that simulates* F.

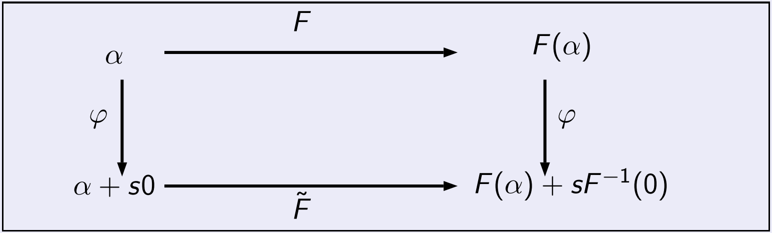
Proof.


$\alpha + s\beta$
$\tilde{F}$
$F(\alpha) + sF^{-1}(\beta)$
H
H
$s\alpha + \beta$
$\tilde{F}$
$F^{-1}(\beta) + sF(\alpha)$


α
F
$F(\alpha)$
φ
φ
$\alpha + s0$
$\tilde{F}$
$F(\alpha) + sF^{-1}(0)$

Proof.





Corollary

There exists a NCTSCA Turing universal.

Morita's extension

Theorem

Given a reversible CA F (partitioned)(one way)

$$\Downarrow$$

There exists $\overline{F}$ a RNCCA that simulate F (neighborhood of size 4).

Morita's extension

Theorem

Given a reversible CA F (partitioned)(neighborhood of size n)

$$\Downarrow$$

There exists $\overline{F}$ a RNCCA that simulate F (neighborhood of size $2n$).

Is another extension of Morita's work possible?

Question

Given a ~~reversible~~ TSCA F (partitioned)(neighborhood of size n)

$$\Downarrow$$

There exists $\overline{F}$ a TSNCCA that simulates F???.

Intrinsic Time-Symmetry

A partitioned time-symmetric CA $F \in \mathrm{End}(Q^{\mathbb{Z}})$ is **intrinsically time-symmetric** (ITS) if its involution is also partitioned.

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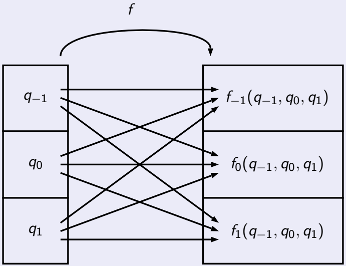
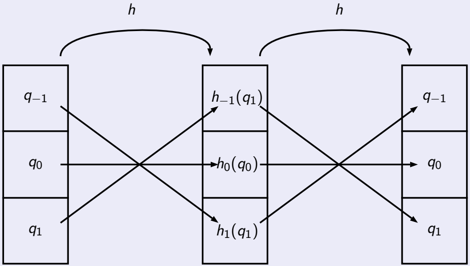
Proposition

H is a RPCA and $H \circ H = Id_{Q^{\mathbb{Z}}} \iff h_i : Q_{-i} \to Q_i$ and $h_i^{-1} = h_{-i}$



Regular PCA



Partitioned involution

Intrinsic Time-Symmetry

Theorem

$$F \in End(Q^{\mathbb{Z}}) \text{ is ITS with involution } H$$

$$\Updownarrow$$

$$f \circ h = h \circ f^{-1} \text{ and } h_i : Q_{-i} \to Q_i \text{ and } h_i^{-1} = h_{-i}$$

Theorem

Given a ITSCA F (neighborhood of size n)

$$\Downarrow$$

There exists $\overline{F}$ *a TSNCCA that simulates F (neighborhood of size 2n).*

i.e. if F is ITSCA then the Morita's simulation preserves the TS.

Thanks