Explanation of the shape of the universal curve of the earthquake recurrence time distributions by means of a cellular automaton model

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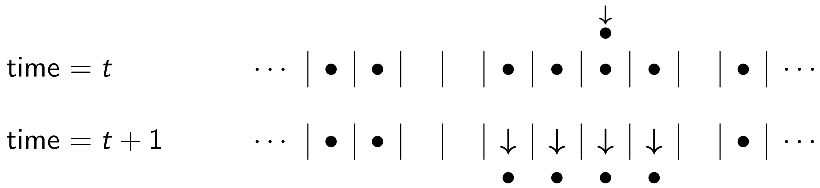
Contents

Definition of Random Domino Automaton

Geometry: N cells in line (circle).

- Stress increases with constant rate in time $\rightarrow$ energy portion "$\bullet$" is added to ramdomly chosen cell (one in a time step)
- Stress exceed some treshold $\rightarrow$ relaxation (avalanche).

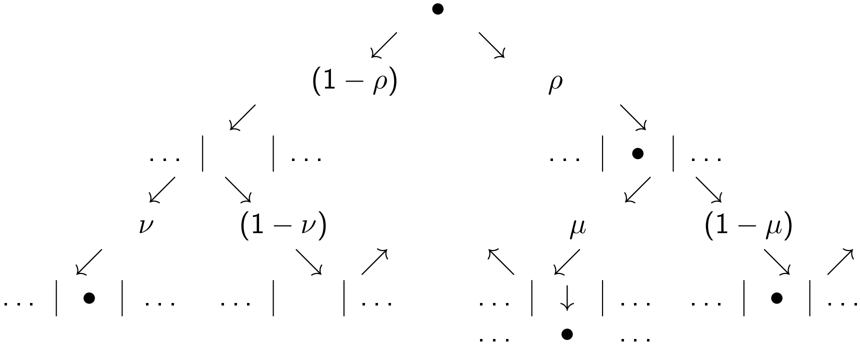


Evolution rule

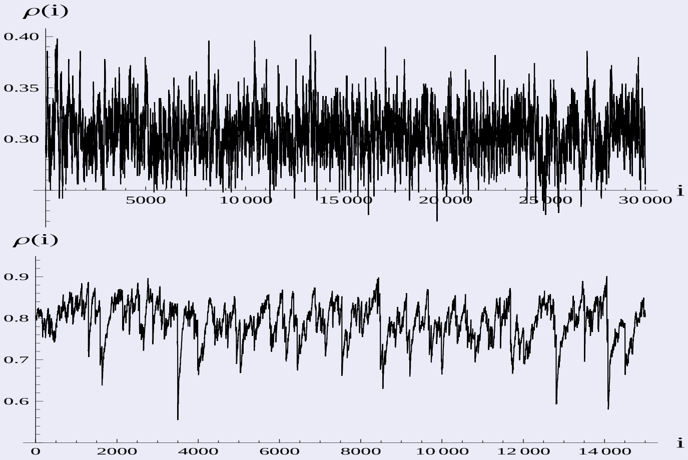
- If a chosen cell is empty, it may become occupied (ν)
- If a chosen cell is occupied, an avalanche may occur ($\mu = \mu(i)$)

★ RDA as a forest-fire model (Drossel-Schwabl model, 1992).

Rebound Parameters



Time Series



Simulation of time series of density $\rho(i)$ for $\mu/\nu = 1$ ($N = 500$) and $\mu(k)/\nu = \frac{0.25}{k}$ ($N = 4000$). Parameter i numbers avalanches.

Equations for a distribution of clusters

Equations for a statistically stationary distribution of sizes of clusters (variables $\{n_1^0, n_1, n_2, \ldots\}$)

$$N = \sum_{i\geq 1} n_i i(\frac{\nu_i}{\nu} i + 1)$$

$$n_1 = \frac{1}{\frac{\mu_1}{\nu} + 2} \left((1-\rho)N - 2n + n_1^0\right)$$

$$n_2 = \frac{2}{2\frac{\mu_2}{\nu} + 2} \left(1 - \frac{n_1^0}{n}\right) n_1$$

$$n_i = \frac{1}{\frac{\mu_i}{\nu} i + 2} \left(2n_{i-1} \left(1 - \frac{n_1^0}{n}\right) + n_1^0 \sum_{k=1}^{i-2} \frac{n_k n_{i-1-k}}{n^2}\right) \qquad i \geq 3,$$

where $n = \sum_{i\geq 1} n_i$ and $\rho = \frac{1}{N} \sum_{i\geq 1} i n_i$.

One-to-one correspondence

Given rebound parameters μ_i/ν determine system's dynamics and distribution of clusters n_i.

Opposite: for given n_i (hence also n, ρ and n_1^0) one may calculate rebound parameters μ_i/ν. For $i \geq 3$ they are given by

$$\frac{\mu_i}{\nu} = \frac{1}{i}\left(2\left(\frac{n_{i-1}}{n_i} - 1\right) + \frac{n_1^0}{n_i n^2}\sum_{k=1}^{i-2} n_k n_{i-1-k}\right).$$

This procedure is valid a priori for arbitrary distribution of clusters n_i.

A one-to-one correspondence $\mu_i = f(n_i)$ allows to establish a relation between parameters μ_i and distribution of avalanches w_i. ($w_i = \frac{\mu_i n_i i}{\sum \mu_i n_i i}$, więc $w_i = c\mu_i f^{-1}(\mu_i) i$, for some constant c.)

Moments of cluster's distribution

Moment m_γ of order γ and **weighted** moment $\widehat{m}_\gamma$ of order γ

$$m_\gamma = \frac{1}{N}\sum_{i\geq 1} n_i i^\gamma \qquad \widehat{m}_\gamma = \frac{1}{N}\sum_{i\geq 1} \frac{\mu_i}{\nu} n_i i^\gamma.$$

Equations for moments

$$\begin{aligned}\widehat{m}_{z+1} &= 1 - m_1 - 2m_0 + 2\sum_{k=0}^{z-1}\binom{z}{k} m_k + \\ &+ \frac{2}{3m_0 + 2\widehat{m}_1}\sum_{l,p=1}^{l+p\leq z}\binom{z}{l+p}\binom{l+p}{l} m_l m_p.\end{aligned}$$

Interpretations of moments

- density $\rho = m_1$
- number of clusters (normalized) $\frac{n}{N} = m_0$

- the moments equation for $z = 0$: $\widehat{m}_1 = 1 - m_1 - 2m_0$
  — balance equation for the number of clusters
- the moments equation for $z = 1$: $\widehat{m}_2 = 1 - m_1$,
  — balance equation for density

- average size of cluster $< i >= \frac{\sum_{i\geq 1} n_i i}{\sum_{i\geq 1} n_i} = \frac{m_1}{m_0}$,
- average avalanche size $< w >= \frac{\sum_{i\geq 1} \mu_i n_i i^2}{\sum_{i\geq 1} \mu_i n_i i} = \frac{\widehat{m}_2}{\widehat{m}_1} = \frac{1-m_1}{1-m_1-2m_0}$.

Three-level description of RDA

Kinetic Theory of Gases

1. Microscopic level: Newton's equations for collisions of gas particles.

RDA Model

1. Microscopic level: geometry and evolution rule for the automaton.

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2. Meoscopic level: velocity distribution function, Boltzman equation.
3. Macroscopic level: moments of velocity distr. function (density of gas, macroscopic velocities, temperature), continuity, Navier-Stokes and heat transport eqs.

RDA Model

1. Microscopic level: geometry and evolution rule for the automaton.
2. Mesoscopic level: equations for clusters' size distribution function.
3. Macroscopic level: equations for moments of clusters' size distribution function, also respective Ito equation describing an evolution of density of a system.

Equations for special case $\mu_i = \delta/i$

Equations for n_i distribution for $\mu_i = \delta/i$ $(\theta = \delta/\nu)$

$$c_{m+2} = c_{m+1} + \sum_{k=0}^{m} c_k c_{m-k} \quad (m \geq 0), \qquad c_0 = c_1 = \frac{1 + \frac{3}{2}\theta + \theta^2}{1 + 4\theta + 4\theta^2}.$$

EXACT solution for arbitrary θ $(m \geq 0)$

$$c_m = \frac{1}{2} \sum_{j=0}^{[\frac{m+2}{2}]} \frac{(2c_0 - \frac{1}{2})^j}{(m-j+2)2^{m-j}} \binom{2(m-j)+1}{m-j+1} \binom{m-j+2}{j}$$

Asymptotics for $\theta \longrightarrow 0$, where $N\theta = const$

$$n_{i+1} \sim \frac{1}{i^{\frac{3}{2}}}.$$

Motzkin numbers

Combinatorial paths' counting, Catalan numbers

The above recurrence is Motzkin numbers recurrence.
It belongs to family of famous Catalan recurrence (its order is 2, not 1) and may be solved using generating function method.

Motzkin number M_n

A number of Motzkin n-paths: paths from $(0,0)$ to $(n,0)$ on lattice $n \times n$ using steps $U=(1,1)$, $F=(1,0)$ and $D=(1,-1)$.

For limit case $\theta = 0$, one has $c_0 = c_1 = 1$, and RDA reduces to the recurrence giving Motzkin numbers:

$$1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, \ldots$$

Thus a new interpretation of Motzkin numbers is accomplished.

Distribution of waiting times for seismicity

Probability density of recurrence times of earthquakes $D(\tau; M_c) = Prob[\tau < \text{rec. time} < \tau + d\tau]/\tau$ for various regions and magnitude ranges is described by some universal curve f: namely $D(\tau; M_c) = R(M_c)f(R(M_c)\tau)$, if time is scalled by average activity $R(M_c)$ of the region (Corral 2004).

- no distinction between foreshocks, mainshocks and aftershocks
- ignore tectonic properties of Earth's crust
- choose the region and minimal magnitude M_c
- point process characterised by occurence time t_i
- recurrence (waiting) time: $\tau_i = t_i - t_{i-1}$

Funct. f is approximable by generalized gamma distr.

$$f^{fit}(\theta) = C\theta^{\gamma-1}\exp(-\frac{\theta^\delta}{\beta}),$$

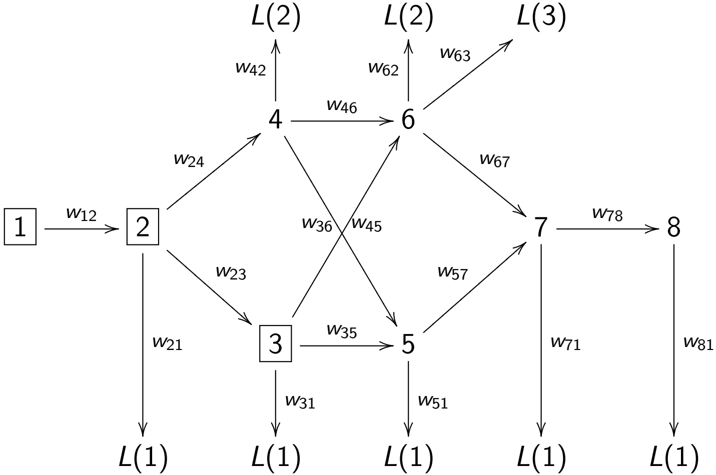
where $\gamma = 0.67 \pm 0.005$, $\beta = 1.58 \pm 0.15$, $\delta = 0.98 \pm 0.05$, $C = 0.5 \pm 0.1$, $\delta \approx 1$ and $\theta = R\tau$ dimensionless recurrence time.

States of RDA for $N = 5$

state label	example	multiplicity
$\boxed{1}$	$\hookrightarrow \mid \; \mid \; \mid \; \mid \; \mid \; \mid \hookleftarrow$	1
$\boxed{2}$	$\hookrightarrow \mid \; \mid \; \mid \; \mid \; \mid \bullet \mid \hookleftarrow$	5
$\boxed{3}$	$\hookrightarrow \mid \; \mid \; \mid \; \mid \bullet \mid \bullet \mid \hookleftarrow$	5
4	$\hookrightarrow \mid \; \mid \; \mid \bullet \mid \; \mid \bullet \mid \hookleftarrow$	5
5	$\hookrightarrow \mid \; \mid \; \mid \bullet \mid \bullet \mid \bullet \mid \hookleftarrow$	5
6	$\hookrightarrow \mid \; \mid \bullet \mid \; \mid \bullet \mid \bullet \mid \hookleftarrow$	5
7	$\hookrightarrow \mid \; \mid \bullet \mid \bullet \mid \bullet \mid \bullet \mid \hookleftarrow$	5
8	$\hookrightarrow \mid \bullet \mid \bullet \mid \bullet \mid \bullet \mid \bullet \mid \hookleftarrow$	1

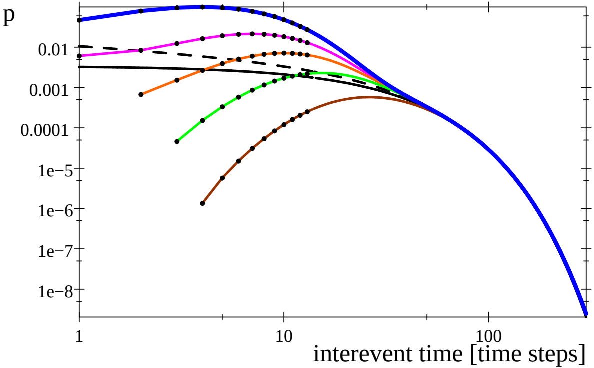
States accessible after avalanches are boxed.

State diagram for RDA of size $N = 5$



Possible transitions with their weights w_{ij}.
Avalanches are indicated by symbol 'L'.

Universal Corral's curve from RDA for $N = 5$



Uppermost line — avalanches of all sizes, line below — of size bigger than 1, next line below — of size bigger than 2 and so on.

Dashed line — fitted gamma distribution $ay^{(\gamma-1)}e^{-\frac{y}{b}}$;
solid line — fitted exponential distribution $a'e^{-\frac{y}{b'}}$.

Summary

Presented results

- three-level description of simple earthquakes model
- solution of inverse problem
- inverse-power distributions, exponential and others
- links to combinatorics
- mechanism for generating waiting times of earthquakes

Other ivestigated topics

- deviations from statistically stationary state
- derivation of respective Ito eqyuations
- finite (small size) version of RDA
- correlations, creation of heavy-tailed distribution
- RDA as a markov chain