Equicontinuity points and eigenvalues of cellular automata

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Motivations / Questions

- M1 : Shift action on sturmian words is conjugate to an irrational rotation.
- M2 :Equicontinuous CA are periodic.
- Q1: Conditions of existence of eigenvalues?
- Q2 : Is it possible to have irrational ones ?

Dynamical systems / Ergodic theory

- Endowed with the distance $d(x, y) = 2^{-n}$ with $n = min\{i \geq 0 : x_i \neq y_i \,\text{or}\, x_{-i} \neq y_{-i}\}$ the set $A^{\mathbb{Z}}$ is a topological compact separated space.
- A cellular automaton is a continuous map $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ commuting with the shift.
- Curtis-Hedlund-Lyndon theorem (local rule such that $F(x)_i = f(x_{i-r}, ..., x_i, ...x_{i+r})$). The integer r is called the radius of the cellular automaton.
- Endowed with the sigma-algebra on $A^{\mathbb{Z}}$ generated by all cylinder sets and the uniform measure $\left(A^{\mathbb{Z}}, \mathbb{B}, F, \mu\right)$ is a measurable space.
- The uniform measure is invariant if and only if the cellular automaton is surjective.

Periodicity

- A point x is said periodic if there exists $p > 0$ with $F^p(x) = x$. The least p with this property is called the period of x.
- A point x is eventually periodic if $F^m(x)$ is periodic for some $m \geq 0$.
- By commutation with the shift every shift-periodic point is $F-$eventually periodic and the set of eventually periodic points is dense.

Density of periodic points

- Closing CA (Boyle & Kitchen) (1999).
- CA with equicontinuous points (Blanchard & Tisseur) (2000).
- CA with almost equicontinuous points (Tisseur) (2008).

Kurka's classification

- A point x is said equicontinuous if for any $\epsilon > 0$ there exist a $\delta > 0$ such that

$$\forall y : d(x, y) < \delta, \forall n \geq 0, d(F^n(y), F^n(x)) < \epsilon.$$

- We say that a cellular automaton is equicontinuous if every point $x \in A^{\mathbb{Z}}$ is an equicontinuous point.
- A cellular automaton is sensitive if there is no equicontinuous points.
- A word w of length s is an s-blocking word for F if there exists $p \in [0, |u| - s]$ such that for any $x, y \in [w]$ we have $F^n(x)(p, p+s) = F^n(y)(p, p+s)$ for all $n \geq 0$.

Gilman's classification

- Based on the Wolfram's work Gilman introduced a classification using Bernoulli measures which are not necessarily invariant.
- Tisseur extends the Gilman's classification to any shift ergodic measure and gives an example of a cellular automaton with an invariant measure which have almost equicontinuous points but without equicontinuous points.

Gilman's classification

Definition

Let F be a cellular automaton and $I = [i_1, i_2]$ a finite interval of $\mathbb{Z}$. For $x \in A^{\mathbb{Z}}$. We define $B_{[i_1,i_2]}(x)$ by :

$$B_I(x) = \left\{ y \in A^{\mathbb{Z}}, \forall j : F^j(x)(i_1, i_2) = F^j(y)(i_1, i_2) \right\}.$$

For any interval I the relation $\mathfrak{R}$ defined by $x \mathfrak{R} y$ if and only if $\forall j : F^j(x)(i_1, i_2) = F^j(y)(i_1, i_2)$ is an equivalence relation and the sets $B_I(x)$ are the equivalence classes.

Gilman's classification

Definition

Let (F,μ) a cellular automaton equipped with a shift ergodic measure μ, a point x is $\mu-$equicontinuous if for any $m>0$ we have :

$$\lim_{n\to\infty}\frac{\mu\left(\left[x\left(-n,n\right)\right]\cap B_{[-m,m]}\left(x\right)\right)}{\mu\left(\left[x\left(-n,n\right)\right]\right)}=1.$$

We say that F is almost expansive if there exist $m>0$ such that for all $x\in A^{\mathbb{Z}}:\mu\left(B_{[-m,m]}\left(x\right)\right)=0$.

Definition

Let (F,μ) denote a cellular automaton equipped with a shift ergodic measure μ. Define classes of cellular automata as follow :

1- $(F,\mu)\in\mathcal{A}$ if F is equicontinuous at some $x\in A^{\mathbb{Z}}$.

2- $(F,\mu)\in\mathcal{B}$ if F is almost equicontinuous at some $x\in A^{\mathbb{Z}}$ but $F\notin\mathcal{A}$.

3- $(F,\mu)\in\mathcal{C}$ if F is almost expansive.

Eigenvalues

- Let F be a CA and $g : A^{\mathbb{Z}} \to \mathbb{C}$ a continuous nonzero function; We say that g is a continuous (or topological) eigenfunction of F associated to the continuous eigenvalue $\lambda \in \mathbb{C}$, if:$g \circ F = \lambda . g$.
- Let (F, ν) be a cellular automaton where ν is an invariant measure. We say that the function $g \in L^2_\mu$ is a measurable eigenfunction associated to the measurable eigenvalue $\lambda \in \mathbb{C}$ if $g \circ F = \lambda . g\ ae$.
- Each eigenvalue be an element of the unit circle i.e of the form $e^{2I\pi\alpha}$.
- We will say that an eigenvalue is rational if $\alpha \in \mathbb{Q}$ and irrational otherwise.

Ergodic / Weakly mixing CA

- A cellular automaton is ergodic if there is no $F-$invariant subset of positive measure.
- It is said weakly mixing if $F \times F$ is ergodic.
- A cellular automaton is ergodic iff any eigenfunction is of constant module.
- A cellular automaton is weakly mixing iff it admits 1 as unique eigenvalue and that all eigenfunctions are constant.

Example 1

Example

Circular CA .

We consider the alphabet $\mathbb{A} = \{0, 1, ...N - 1\}$ and the circular cellular automaton defined by :

$$\{(F(x))_i = (x_i + 1) \bmod N$$

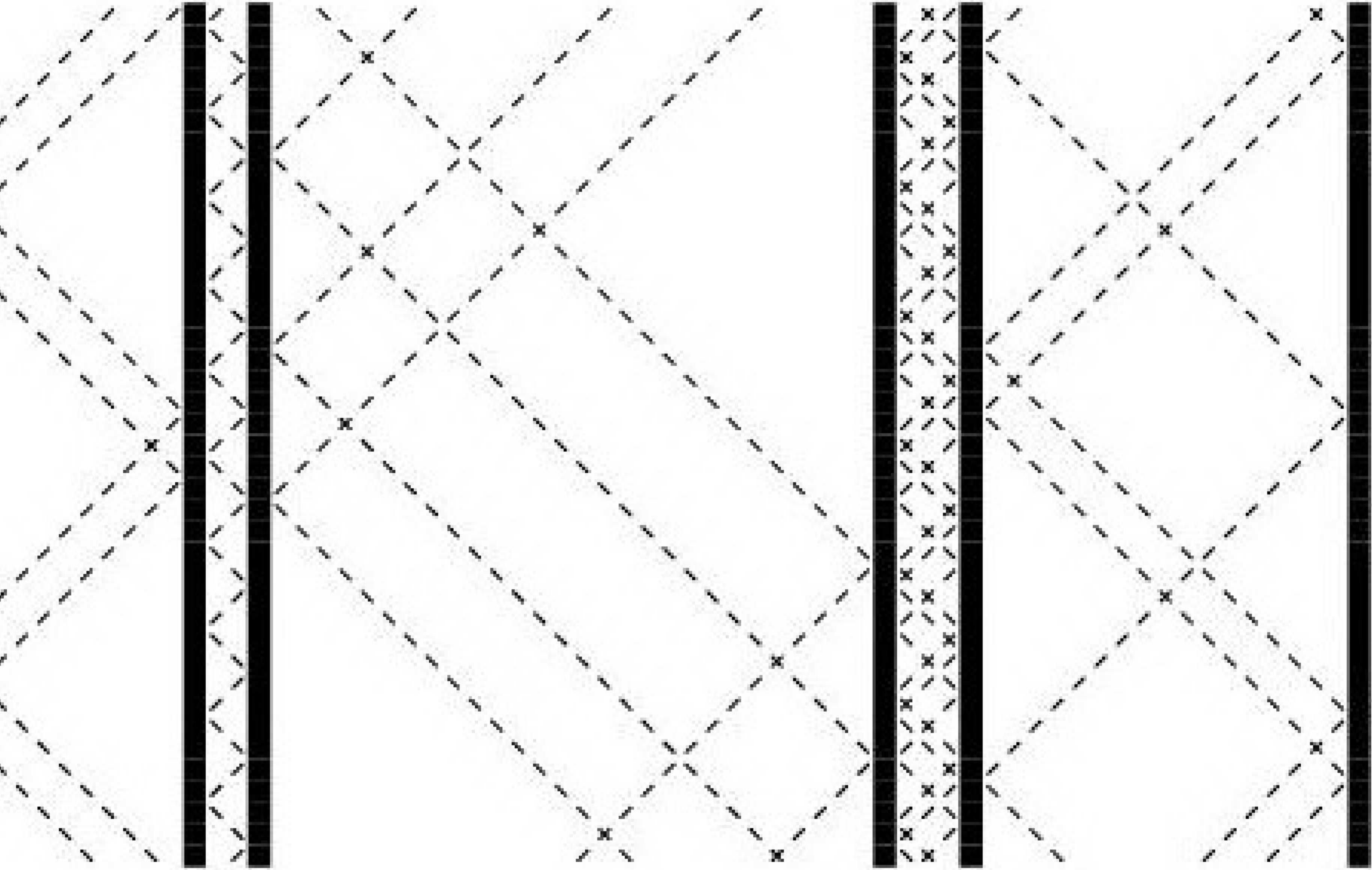
This cellular automaton has the eigenvalues $\left(e^{2I\pi\left(\frac{p}{N}\right)}\right)_{1 \leq p \leq N-1}$ associated to eigenfunctions $\left(g(x) = e^{2I\pi\left(\frac{p}{N}\right)x_0}\right)_{1 \leq p \leq N-1}$.

Example 2

Example

Kůrka's vehicle $\left(\mathbf{5}^{\mathbb{Z}}, B\right)$ where $\mathbf{5} = \{0, l, r, lr, w\}$
The dynamics can be described in terms of the movements of three types of particles :

- A wall (w).An empty cell (0)
- A particle (l) that goes to the left, when it hits a wall (w) it changes to a r particle that goes to the right.
- The particle lr corresponds to a cell occupied by both l and r.



Example 3

Example

Sensitive CA

Product of the shift and any CA with some equicontinuous point.

Topological (Preliminary) results.

- Due to the density of shift periodic points hence of eventually periodic points there is no continuous irrational eigenvalues.
- A direct consequence of the continuity of the eigenfunction is that if a CA has rational eigenvalues then it has a factor which is a circular cellular automaton.

Topological (Preliminary) results.

Proposition

Let F a surjective cellular automaton with equicontinuous points but without being equicontinuous; then it has an infinity of rational eigenvalues.

Proof.

[Main idea]

Existence of a rational eigenvalue : Just read what happens between two blocking words.

Infinity of eigenvalues.

i) Suppose that the eigenvalues are in finite number : there exist s periods p_i associated to each eigenvalue.

ii) For any x in $A^{\mathbb{Z}}$ and any $n > 0$ the points $(wx(-n, n)w)^{\infty}$ are periodic for F Hence each cylinder $[wx(-n, n)w]$ can be associated to an eigenvalue.

iii) Taking the common period + density of periodic points $\Rightarrow F$

Ergodic results

Proposition

Let (F, ν) be a surjective cellular automaton with $\nu-$equicontinuous points then F has strictly ergodic rational eigenvalues.

Proof.

Let x be a ν-equicontinuous point: We have a measurable wall w.
Let u be a finite word.
By incorporating the word uw in y between the words w and u we obtain a new element $y^{(1)}$ containing two occurrences of the word uw (see Fig.1). □

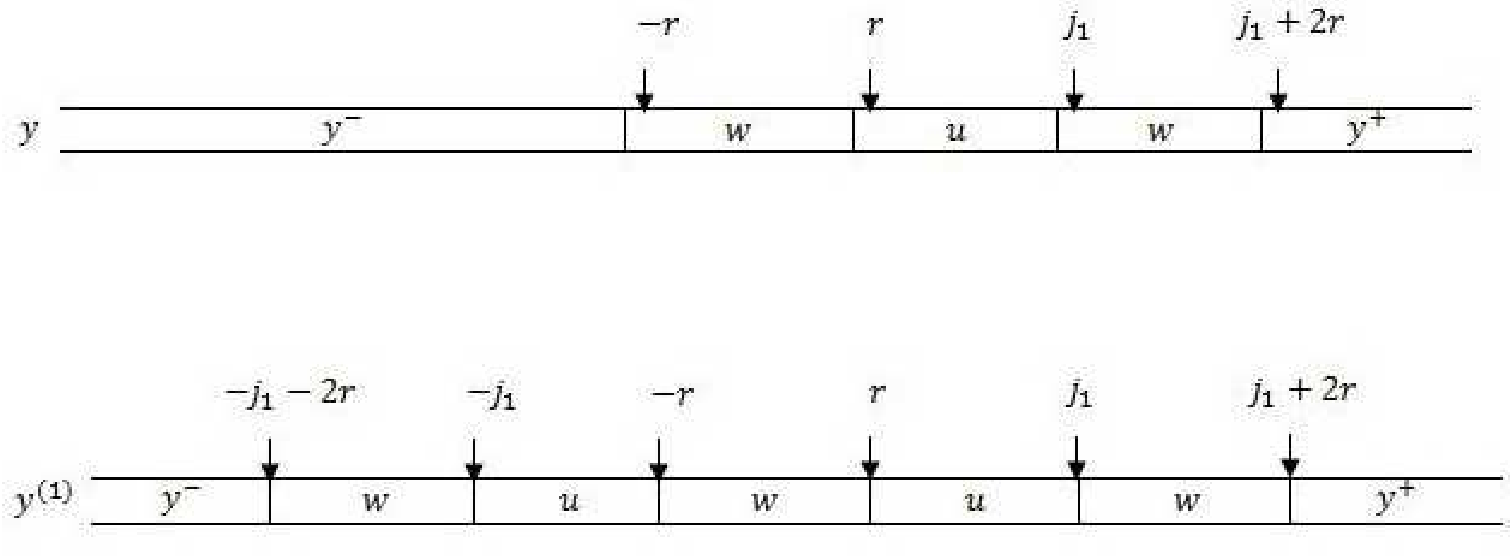


Fig 1 : Recursive construction of a periodic element.

Ergodic results

Definition

Let (F, ν) be a cellular automaton, We define the family of measurable sets Q_n by :

$$Q_n = \left\{ x : \left(F^i(x)(-n, n) \right)_{i \in \mathbb{N}} \text{ eventualy periodic} \right\}.$$

Lemma

Let (F, ν) be a surjective cellular automaton; if $\nu(Q_n) = 1$ for every n then F cannot have any measurable irrational eigenvalue.

Main Idea : The periodic behaviour makes points too close to simulate an irrational rotation.

Ergodic results

Proposition

Let (F, ν) be a surjective cellular automaton with $\nu-$equicontinuous points then F can't have irrational measurable eigenvalue.

Proof.

Use same technique as in proof of rational eigenvalue to show that $\nu\left(Q_n\right) = 1$ for every arbitrary value of n □

Thank you