Group-Walking Automata

Ville Salo[1] Ilkka Törmä[2]

[1]Center for Mathematical Modeling
University of Chile

[2]TUCS – Turku Centre for Computer Science
University of Turku

AUTOMATA 2015
Turku, Finland

Different Kinds of Finite Automata

- One-way finite automata on words
- Two-way finite automata on words
- Four-way finite automata on pictures
- Automata that walk on trees or *graphs*

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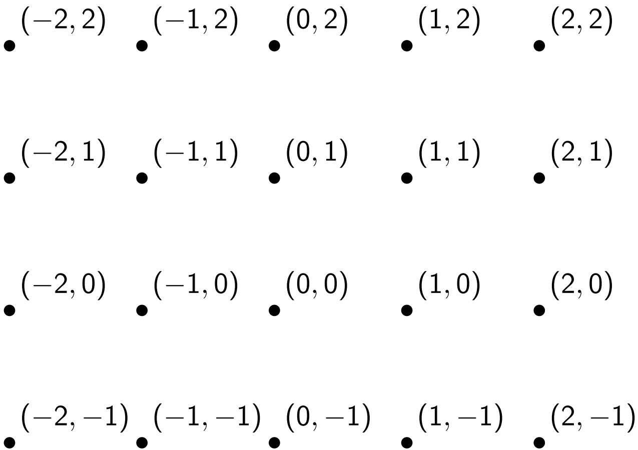
- Automata that walk on trees or *graphs*
- In this presentation: Cayley graphs of finitely generated infinite groups

Cayley Graphs

- Take a group $(G, \cdot)$ generated by $S = \{g_1, \ldots, g_n\}$
- We usually assume S *symmetric*: if $g \in S$ then $g^{-1} \in S$
- The *Cayley graph of G* has vertex set G and edges $g \to g \cdot g_i$ for $i = 1, \ldots, n$
- It is infinite if G is, and every vertex has degree n

Cayley Graphs

Example: the discrete plane $\mathbb{Z}^2$



Cayley Graphs

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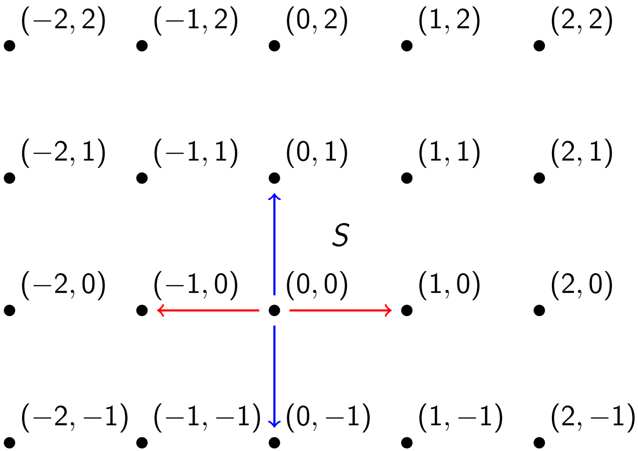

Cayley Graphs

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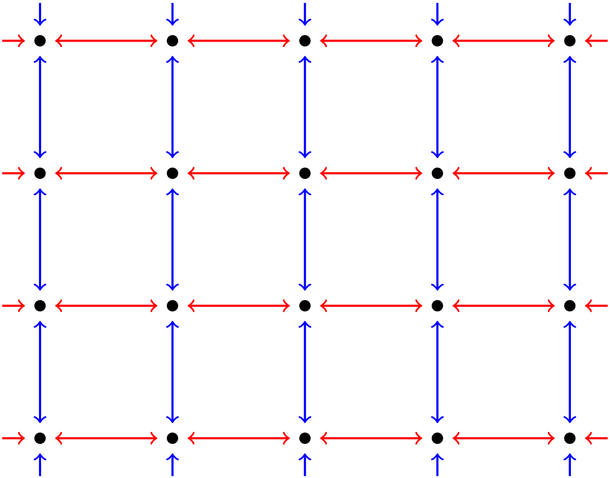
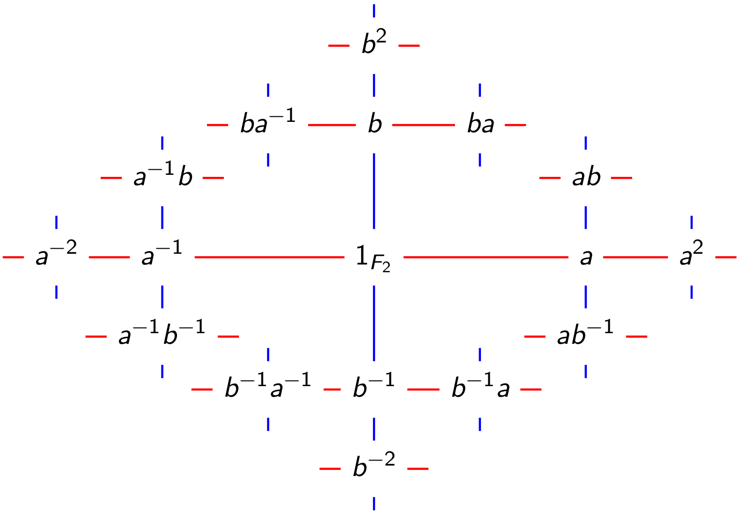


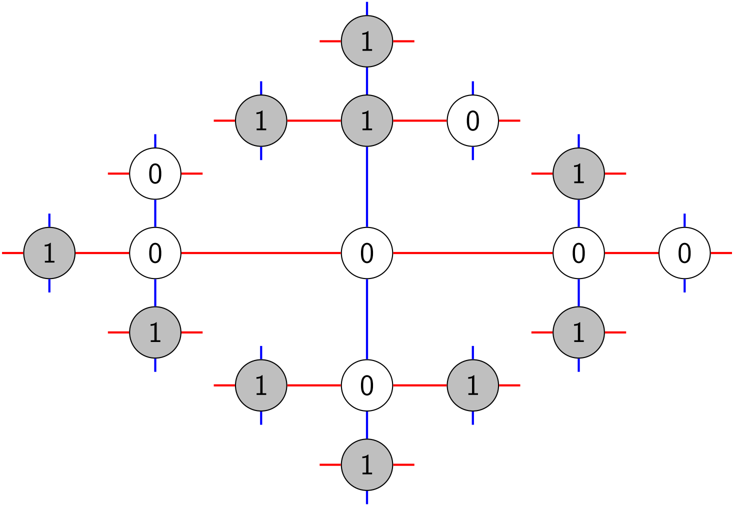
Cayley Graphs

Example: the free group $F_2 = \langle a, b \rangle$



Cayley Graphs

Example: the free group $F_2 = \langle a, b \rangle$, with vertex coloring



Group-Walking Automata

Definition (Group-Walking Automaton)

- A *group-walking automaton* is a multi-head finite automaton that walks on the Cayley graph of G
- It recognizes vertex colorings of G
- A coloring is *rejected* if, started from *some* single vertex in *some* initial states, the heads eventually return together and enter a rejecting state
- The class of all sets of colorings accepted by automata (with k heads) is $\mathcal{S}(G)$ ($\mathcal{S}(G, k)$)

Group-Walking Automata

The heads of group-walking automata *can*:

- Take synchronized steps of variable length along the edges of the graph
- Read the colors of the vertices, and the states of other heads on the same vertex
- Change their internal state

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The heads of group-walking automata *can*:

- Take synchronized steps of variable length along the edges of the graph
- Read the colors of the vertices, and the states of other heads on the same vertex
- Change their internal state

They *cannot*:

- Take arbitrarily long steps
- Read the colors of faraway vertices, or the states of faraway heads
- Change the colors of the vertices

Group-Walking Automata

- Every set of colorings in $\mathcal{S}(G)$ is defined by *forbidden finite patterns*: it is a *G-subshift*
- Which subshifts are in $\mathcal{S}(G, k)$ for different k? How does this depend on G?
- On *finite* words, trees and grids, we have infinite hierarchies: adding more heads increases the model's power

Previous Results

V.S. & I.T.: *Plane-Walking Automata* (presented at AUTOMATA 2014, Himeji, Japan)

- $\mathcal{S}(\mathbb{Z}^d, 3) = \mathcal{S}(\mathbb{Z}^d)$, contains exactly the Π^0_1 subshifts
- $\mathcal{S}(\mathbb{Z}^d, 1) \subsetneq \mathcal{S}(\mathbb{Z}^d, 2) \subseteq \mathcal{S}(\mathbb{Z}^d, 3)$; second inclusion strict for $d \geq 3$, unknown for $d \leq 2$

Π^0_1 (or *effectively closed*) subshifts have computable sets of forbidden patterns

Our Results

Theorem (Three Heads)

If G is not torsion, then $\mathcal{S}(G,3)$ contains all Π^0_1 subshifts; if G also has decidable word problem, then the classes coincide and $\mathcal{S}(G) = \mathcal{S}(G,3)$

- *Proof idea*: Simulate a certain type of Turing-universal counter machine on the Cayley graph using 3 heads

Our Results

Theorem (Characterization of Torsion Groups)

For all G, every subshift in $\mathcal{S}(G)$ is intrinsically Π^0_1; the classes coincide iff $\mathcal{S}(G) = \mathcal{S}(G, 4)$ iff G is not torsion

- Intrinsically Π^0_1 subshifts have sets of forbidden patterns computable from the word problem of G
- G is a torsion group if every $g \in G$ has finite order: $g^n = 1_G$ for some $n > 0$
- *Proof idea*: In non-torsion case, decide word problem with 4 heads; walking on torsion group results in a loop

Three Heads

Definition (Arithmetical Program)

An *arithmetical program* is a one-head finite automaton equipped with an unbounded *counter*, and it can

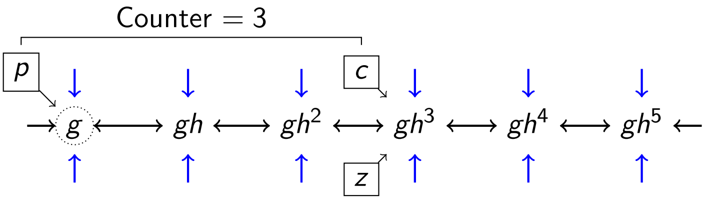
- walk on the Cayley graph of G and read its colors
- increment and decrement the counter
- check the remainder of the counter modulo fixed positive integers, and whether it is 0
- multiply or divide the counter by fixed positive integers
- reject the coloring at any point

Arithmetical programs recognize exactly the Π_1^0 subshifts

Three Heads

Simulating an arithmetical program with 3 heads:

- $h \in G$ has infinite order
- counter value is distance between p and c in powers of h
- multiplication, division and movement are implemented using synchronized signals and the z head



Four Heads

- With a fourth head we can solve the word problem of G
- For a product of generators $g_{i_1} \cdots g_{i_n}$, leave the fourth head behind and walk along the g_{i_j} with the others
- If they return to the fourth head, $g_{i_1} \cdots g_{i_n} = 1_G$
- Then we can recognize all intrinsically Π^0_1 subshifts

Torsion Groups

- Finitely generated infinite torsion groups are hard to construct (*Burnside Problem*)
- In a torsion group, a single head walking in any direction ends up in a loop

Torsion Groups

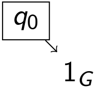
Lemma (Navigation on Torsion Groups)

Let G be a torsion group. There exists $d_G : \mathbb{N}^3 \to \mathbb{N}$ such that for all k-head q-state automata that can take steps of length r, no head can move more than $d_G(k, q, r)$ steps away from its starting point on the all-0 coloring of G.

Proof by induction on k

Torsion Groups

First case: one head ($k = 1$)



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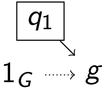

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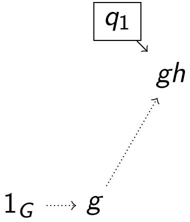

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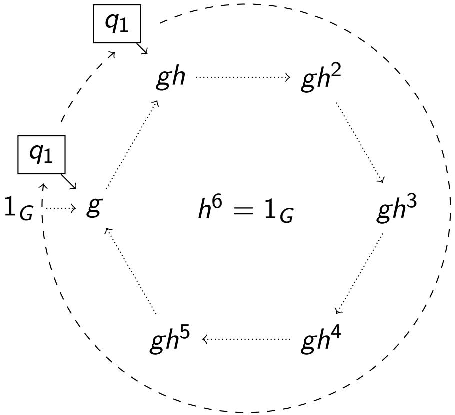
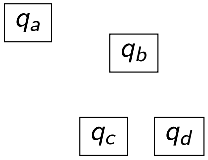


Torsion Groups

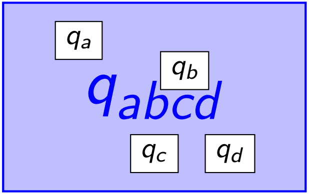
Second case: all heads stay close to each other



Torsion Groups

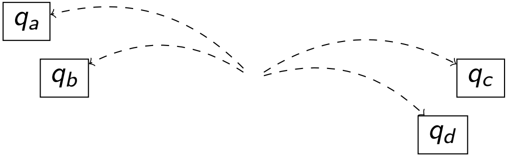
Second case: all heads stay close to each other

Combine into one head with larger state set, and reduce to the $k = 1$ case



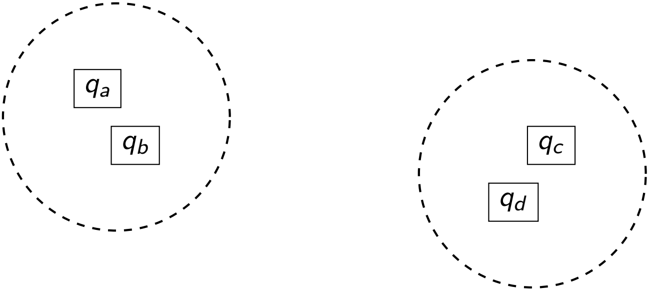
Torsion Groups

Third case: some heads travel far from others



Torsion Groups

Third case: some heads travel far from others
Apply induction hypothesis to every separated group: they never communicate again and travel a bounded distance



Torsion Groups

Corollary

If G is a torsion group, then the subshift of colorings $x : G \to \{0,1\}$ with $\#\{g \in G \mid x_g = 1\} \leq 1$ is not in $\mathcal{S}(G)$

In particular, $\mathcal{S}(G)$ does not contain all intrinsically Π^0_1 subshifts

Future Work

We showed that $\mathcal{S}(G) = \mathcal{S}(G, 4)$ if G is not a torsion group, and $\mathcal{S}(G) = \mathcal{S}(G, 3)$ if G also has a decidable word problem

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Conjecture (4 Heads Better than 3)

There exists a non-torsion group G such that $\mathcal{S}(G, 3) \subsetneq \mathcal{S}(G, 4)$

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Conjecture (4 Heads Better than 3)

There exists a non-torsion group G such that $\mathcal{S}(G, 3) \subsetneq \mathcal{S}(G, 4)$

Conjecture (Infinite Hierarchy)

There exists a torsion group G such that the hierarchy $\mathcal{S}(G, k)_{k \geq 1}$ is infinite

The End

Thank you!