# Degenerate hyperbolicity in cellular automata

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# Motivation

- In a linear continuous-time dynamical system $\dot{\mathbf{x}} = A\mathbf{x}$, if $\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$ and $A$ is a real $n \times n$ matrix with all eigenvalues distinct and having negative real parts, $\mathbf{x}(t) \to 0$ exponentially fast as $t \to \infty$.
- The same phenomenon can be observed in nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ (where $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$) in a vicinity of hyperbolic fixed point, as long as the Jacobian matrix of $\mathbf{f}$ evaluated at the fixed point has only distinct eigenvalues with negative real parts.
- Discrete dynamical systems exhibit similar behaviour.

# Degeneracy

If eigenvalues are degenerate (repeated), the convergence is linear-exponential. Example:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} \tag{1}$$

is defined by a matrix which has degenerate (double) eigenvalue $\frac{1}{2}$,

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \left(\frac{1}{2}\right)^n \begin{bmatrix} 1-n & 2n \\ -\frac{n}{2} & 1+n \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \tag{2}$$

# Can we have this in cellular automata?

We will consider 3-state nearest-neighbour CA obtained from elementary binary CA by "lifting" them to 3-states. Let $g : \{0,1\}^3 \to \{0,1\}$ be a local function of elementary CA satisfying $g(0,0,0) = 0$, $g(1,1,1) = 1$, and let $f_g : \{0,1,2\}^3 \to \{0,1,2\}$ be defined by

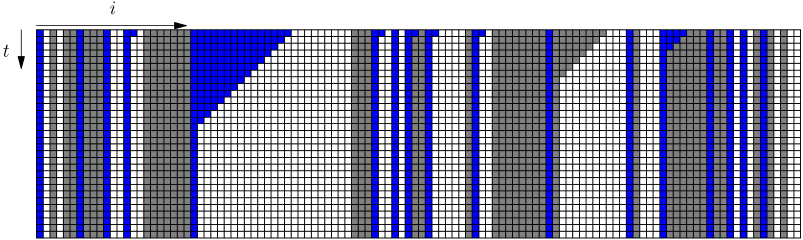
$$f_g(x_1,x_2,x_3) = \begin{cases} g(x_1,x_2,x_3), & x_1,x_2,x_3 \in \{0,1\} \\ 2g\left(\frac{x_1}{2},\frac{x_2}{2},\frac{x_3}{2}\right), & x_1,x_2,x_3 \in \{0,2\} \\ g(x_1-1,x_2-1,x_3-1)+1, & x_1,x_2,x_3 \in \{1,2\} \\ x_2, & \text{otherwise.} \end{cases}$$

# Rule 140

One of the most interesting cases is $f_g$ defined by $g$ being the elementary CA rule 140, given by

$$g(x_1, x_2, x_3) = x_2 - x_1 x_2 + x_1 x_2 x_3$$

for $x_1, x_2, x_3 \in \{0, 1\}$. Rule $f_g$ has the following spatio-temporal pattern:



# Block evolution operator

Let $\mathcal{B}$ be the set off all finite blocks (words). A *block evolution operator* corresponding to $f$ is a mapping $\mathbf{f} : \mathcal{B} \mapsto \mathcal{B}$ defined as follows. Let $a = a_0 a_1 \ldots a_{n-1} \in \mathcal{B}$ where $n \geq 3$. Then

$$\mathbf{f}(a) = \{f(a_i, a_{i+1}, \ldots, a_{i+2})\}_{i=0}^{n-3}.$$

Example: for 3-state rule 140,
$\mathbf{f}^4(202112011110210201) = 120100002,$

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 0 | 2 | 1 | 1 | 2 | 0 | 1 | 1 | 1 | 1 | 0 | 2 | 1 | 0 | 2 | 0 | 1 |
| | 0 | 2 | 1 | 1 | 2 | 0 | 1 | 1 | 1 | 0 | 0 | 2 | 1 | 0 | 2 | 0 | |
| | | 2 | 1 | 1 | 2 | 0 | 1 | 1 | 0 | 0 | 0 | 2 | 1 | 0 | 2 | | |
| | | | 1 | 1 | 2 | 0 | 1 | 0 | 0 | 0 | 0 | 2 | 1 | 0 | | | |
| | | | | 1 | 2 | 0 | 1 | 0 | 0 | 0 | 0 | 2 | 1 | | | | |

# Preimages

A given block $b$ has typically more than one preimage under $\mathbf{f}$. Let the *density polynomial* associated with a string $\mathbf{b} = b_1 b_2 \ldots b_n$ be defined as

$$\Psi_{\mathbf{b}}(p, q, r) = p^{\#_0(\mathbf{b})} q^{\#_1(\mathbf{b})} r^{\#_2(\mathbf{b})}, \tag{3}$$

where $\#_i(\mathbf{b})$ is the number of occurrences of symbol $i$ in $\mathbf{b}$. If $A$ is a set of binary strings, we define density polynomial associated with $A$ as

$$\Psi_A(p, q, r) = \sum_{\mathbf{a} \in A} \Psi_{\mathbf{a}}(p, q, r). \tag{4}$$

# Preimages

One can show that if one starts with a bi-infinite string of symbols drawn from Bernoulli distribution where probabilities of $0, 1$ and $2$ are, respectively, $p, q$ and $r$, then the expected fraction of sites in state $k$ after $n$ iterations of rule $f$ is given by

$$\Psi_{\mathbf{f}^{-n}(k)}(p, q, r)$$

This quantity will be called *density* of symbols $k$ after $n$ iterations of $f$.

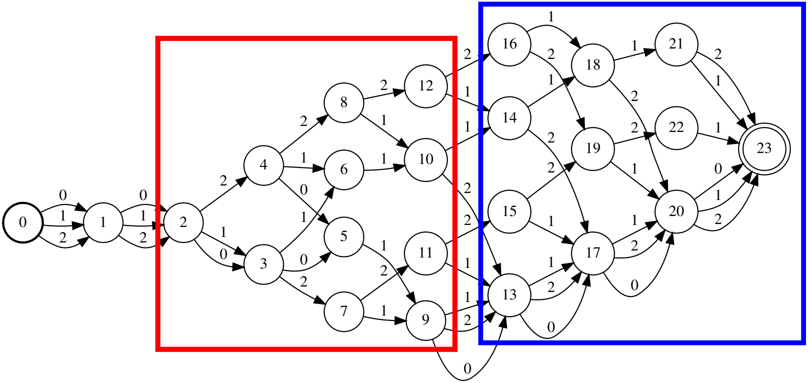
# Preimages of 1 for rule 140

In order to find a "closed form" expression for the density, we need to describe the structure of preimage sets $\mathbf{f}^{-n}(k)$. These sets grow very quickly. For example, for rule 140, $\mathbf{f}^{-4}(1)$ has 6048 elements. Some of them:

000010000
000010001
000010002
000010010
000010011
000010012
000010020
...

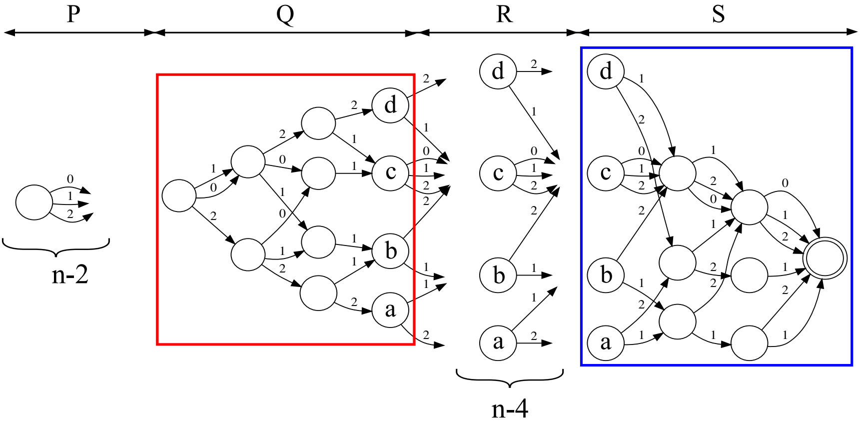
# Preimages of 1 for rule 140

The set $\mathbf{f}^{-4}(1)$ can be described by FSM:



# General FSM

In general, the set $\mathbf{f}^{-n}(1)$ can be described by



# Density polynomial for $r \neq q$

Using the FSM, one can construct density polynomial,

$$\Psi_{\mathbf{f}^{-n}(1)}(p,q,r) = \frac{pq^2\left(-pr+pq+q^2\right)(q\lambda)^n}{\lambda^2\,(p+r)\,(q-r)} + \frac{qr\left(-p^2r+p^2q+pq^2-2\,pqr+r^3-q^2r\right)(r\lambda)^n}{\lambda^2\,(p+q)\,(q-r)} + \frac{q\left(p^3+p^2q+2\,p^2r+pr^2+3\,pqr+r^3+r^2q+q^2r\right)\lambda^{2n}}{\lambda\,(p+r)\,(p+q)},$$

where we used $\lambda = p+q+r$.

# Degenerate case, $r = q$

$$\Psi_{\mathbf{f}^{-n}(1)}(p,q,q) = \frac{pq^3 (n+1)(q\lambda)^n}{\lambda^2 (q+p)} + \frac{q^2 \left(2\,p^3 + 4\,p^2 q + pq^2 - 2\,q^3\right)(q\lambda)^n}{(q+p)^2 \lambda^2} + \frac{\left(p^3 + 3\,p^2 q + 4\,pq^2 + 3\,q^3\right) q\lambda^{2n}}{\lambda (q+p)^2},$$

When $p = 1, q = 1$ and $r = 1$, then $\Psi_{\mathbf{f}^{-n}(1)}(1,1,1)$ counts the number of preimages of 1. This yields a sequence exhibiting linear-exponential growth,

$$\Psi_{\mathbf{f}^{-n}(1)}(1,1,1) = \left(\frac{n}{18} + \frac{7}{36}\right) 3^n + \frac{11}{12} 9^n.$$

# Questions

- What is the meaning of the degeneracy?
- Does it mean that dynamics of rule 140 (or other rules) can be effectively reduced to a finite-dimensional system of difference equations?
- Does it mean that local structure approximation becomes exact at some level for rule 140? Or for other rules?