More Decision Algorithms for Global Properties of 1D CA

Sébastien Autran* Enrico Formenti
Julien Provillard
{autran,formenti,provilla}@i3s.unice.fr

Nice Sophia Antipolis University, I3S UMR 7271, CNRS, France

Objective

- Decision algorithm for surjectivity on finite configurations for 1D CA
- Simple decision algorithm for openess
- Implication diagram between global properties of 1D CA

1D CA

Formally, a (one-dimensional) cellular automaton (1D CA) is a structure

$$\mathcal{A} = \langle r, \Sigma, [-r, r], \delta \rangle$$

- $r \in \mathbb{N}$ — *radius*
- Σ — (finite) set of *states*
- $[-r, r]$ — *neighborhood*
- $\delta \colon \Sigma^{2r+1} \to \Sigma$ — *local function*

A 1D CA is *elementary* (ECA) $\to$ $\Sigma = \{0, 1\}$ and $r = 1$.

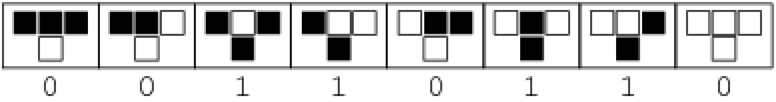
Wolfram number

The *Wolfram number* of a table $\delta : \{0,1\}^{2r+1} \to \{0,1\}$ is given by

$$\sum_{x \in \{0,1\}^{2r+1}} 2^{dec(x)} \cdot \delta(x)$$

where $dec(x)$ is the function which assigns the decimal representation to the binary number x.

Example: rule 54



Global rule

The global rule $G\colon \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ describes the overall behavior of a CA from time step t to $t+1$ for any $t \in \mathbb{N}$:

$$\forall c \in \{0,1\}^{\mathbb{Z}}, \forall i \in \mathbb{Z},\ G(c)_i = \delta(c_{[i-r,i+r]}) \ .$$

Finite and periodic configurations

A configuration c is x-finite for $x \in \Sigma$ if $|\{i \in \mathbb{Z},\ c_i \neq x\}| < \infty$.

Let $\mathcal{F}$ be the set of all x-finite configurations for a fixed $x \in \Sigma$.

A configuration c is spatially periodic if $\exists p \in \mathbb{N}$ *st.* $\sigma^p(c) = c$ where σ is the shift map.

Let $\mathcal{P}$ be the set of all spatially periodic configurations.

G bijective $\Leftrightarrow$ G reversible $\Leftrightarrow$ G injective
$\Leftrightarrow$ $G|_{\mathcal{P}}$ bijective $\Leftrightarrow$ $G|_{\mathcal{P}}$ injective

$\Downarrow$

$G|_{\mathcal{F}}$ bijective
$\Leftrightarrow$ $G|_{\mathcal{F}}$ surjective

$\Downarrow$

$G|_{\mathcal{F}}$ injective $\Leftrightarrow$ $GoE = \emptyset$ $\Leftrightarrow$ G surjective
$\Leftrightarrow$ $G|_{\mathcal{P}}$ surjective

We differentiate surjectivity according to $x \in \Sigma$:

- $G|_{\mathcal{F}_x}$ is the restriction of G to x-finite configurations.
- $G|_{\mathcal{F}_\bigstar} := \bigcup_{x \in \Sigma} G|_{\mathcal{F}_x}$.
- $G|_{\mathcal{F}^2} := G^2|_{\mathcal{F}}$ with $\delta(0\ldots0) = 1$ and $\delta(1\ldots1) = 0$.

Definition

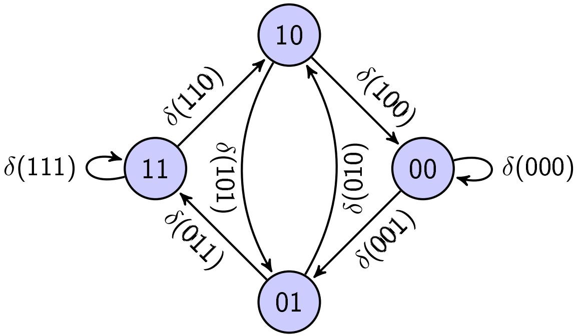
In the case of binary CA, $G|_{\mathcal{F}}$ is *surjective* if $G|_{\mathcal{F}_x}$ is surjective for some $x \in \Sigma$.

Definition

Given two binary strings $x, y \in \{0,1\}^{2r+1}$, the *fusion* $x \odot y$ of x and y is the string $x_0 y$ if $x_i = y_{i-1}$ for $i \in [1, 2r]$, $\perp$ otherwise.

The *De Bruijn* graph B_δ associated with a CA $\langle r, \Sigma, [-r, r], \delta \rangle$ is a structure $\langle V, E, l \rangle$,

- $V = \{0,1\}^{2r}$
- $E = \{(x, y) \in V \times V \mid x \odot y \neq \perp\}$
- $l \colon E \to \{0,1\}$ is a labelling function such that $l(x, y) = \delta(x \odot y)$.

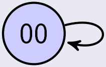


Remark

Given a bi-infinite path $P = \dots p_0, \dots, p_n \dots$ over B_δ, the string $\dots \odot p_0 \odot p_1 \odot \dots \odot p_n \odot \dots$ is a configuration for the CA δ and $l(P) = \dots l(p_0, p_1) \cdot \dots \cdot l(p_{n-1}, p_n) \dots$ is its image.

Remark

$$G({}^{\omega}0^{\omega}) = {}^{\omega}0^{\omega}$$



$$\delta(000) = 0$$

Proposition 1

For $u, v \in \mathcal{F}$, if $G(x^\omega v x^\omega) = x^\omega u x^\omega$, then the configuration has to start and end in vertex $i = \{x\}^{2r}$ with $\delta(i \odot i) = x$ on the associated De Bruijn graph.

Basic idea

Modify the initial and final sets of vertices of B in a new graph B^T in which we trim configurations by x *st.*

$$u \in L(B^T) \Leftrightarrow \exists k, k' \geq 0 : x^k u x^{k'} \in L(B)$$

Construction of B^T from B

Let a De Bruijn graph $B = (\Sigma, I, F)$

- set of states Σ
- set of initial states I
- set of final states F

Let $i = \{x\}^{2r}$
$I = F = \{i\}$ if $\delta(i \odot i) = x$, $\{\perp\}$ otherwise.

$I(y, u) \rightarrow$ the vertex reached starting from vertex y after reading word u as a sequence of labels.

Construction of B^T

$$I^T = \{I(q', x^*), q' \in I\}$$

$$F^T = \{q \in \Sigma, \exists k \geq 0, I(q, x^k) \in F\}$$

$$B^T = (\Sigma, I^T, F^T)$$

Proposition 2

$u \in L(B^T) \Leftrightarrow \exists k, k' \geq 0 : x^k u x^{k'} \in L(B).$

Proof

Proof.

[⇒] ($\Leftarrow$ is obvious from the construction of B^T)

For $\{x\}^{2r} \in I$, $u \in L(B^T) \Rightarrow \exists q \in I^T, \exists q' \in F^T : l(q,u) = q'$

$$\exists k \geq 0 : l(\{x\}^{2r}, x^k) = q,$$

$$\exists k' \geq 0 : l(q', x^{k'}) = \{x\}^{2r} \in F \Rightarrow x^k u x^{k'} \in L(B)$$

□

Test surjectivity on $\mathcal{F}$

Proposition 3

$\forall x \in \Sigma,\ G|_{F_x}$ is surjective *iff* $L(B^T) = \Sigma^*$.

Proof

Proof.

From Proposition 2, we know that if $u \in L(B^T)$, then $\exists k, k' \geq 0 : x^k u x^{k'} \in L(B)$. In other words, it means that for a recognized configuration u in B^T, the first symbol of u is equal to the last symbol of u and is not equal to x. In addition, to be surjective, one need that each configuration has a least one pre-image. Thus, all configurations of B^T should have a pre-image, which is equivalent to test if $L(B^T) = \Sigma^*$. □

Test equality to Σ^* .

Basic idea

Determinize B^T and detect failure during construction.

In the deterministic graph, starting from vertex I^T

- there should be an outgoing edge from current vertex labelled by x for any possible x belonging to Σ
- if it contains a vertex $v \in F^T$, then it is final.

Since we construct the deterministic graph and we want to read each symbol, if the current created vertex is not final or has not an outgoing edge labelled by x, $\forall x \in \Sigma$, then $L(B^T) \neq \Sigma^*$.

Complexity

There are:

- 2^{2r} vertices in B
- 2^{2r} vertices in B^T
- $2^{2^{2r}}$ vertices in the deterministic graph of B^T *(in the worst case)*

Complexity

Our algorithm is exponential in the size of the local rule.

Results for ECA

Wolfram number of ECA			Surjectivity type
	166	180	$G\|_{\mathcal{F}_0}$
	154	210	$G\|_{\mathcal{F}_1}$
170	204	240	$G\|_{\mathcal{F}_\star}$
15	51	85	$G\|_{\mathcal{F}^2}$

In order to test if $G|_{\mathcal{F}^2}$ is surjective, increase radius by 1.

Objective

Complete the implication diagram by adding the openess property.

Definition

Recall that two configurations $x, y \in \{0,1\}^{\mathbb{Z}}$ are *left-asympthotic* (resp., *right-asympthotic*) if there exists $i \in \mathbb{Z}$ such that $x_j = y_j$ for all $j > i$. A CA is *right closing* (resp., *left closing*) if it is injective on left-asympthotic (resp., right-asympthotic) configurations.
A CA is open if and only if it is both left and right closing.

injective $\Rightarrow$ open $\Rightarrow$ right or left closing $\Rightarrow$ surjective

Given B_δ, Sutner creates a graph B_δ^2 where:

- each vertex is a pair of vertices of B_δ
- there is an edge with label i between two vertices of B_δ^2 $\langle x, y\rangle$ and $\langle x', y'\rangle$ if there is an edge with the same label i between x, x' and y, y'

After defining $B_\delta^2::scc$, its condensed graph (in which one vertex represents a strongly connected component - *abbr.* SCC), there are 3 types of vertices in $B_\delta^2::scc$

- a component called Δ, made of the pairs of vertices $\langle v, v\rangle$.
- trivial SCC, which are transient vertices and do not need to be considered (because they do not belong to an infinite configuration).
- non-trivial SCC, called A_i vertices.

Proposition

Consider a 1D CA δ and let B^2_δ::scc be its condensed graph. Then,

1) *δ is **not surjective** iff Δ is not a SCC (i.e. Δ is contained in a SCC with other vertices);*
2) *δ is **surjective** iff Δ is one of the SCC;*
3) *δ is **non-right closing** iff it exists at least a path from Δ to a A_i vertex;*
4) *δ is **non-left closing** iff it exists at least a path from a A_i vertex to Δ;*
5) *δ is **open** iff there is neither a path from a A_i vertex to Δ nor a path from Δ to a A_i vertex (which means that we can neither enter nor leave Δ);*
6) *δ is **injective** iff B^2_δ::scc does not contain another non-trivial SCC other than Δ.*

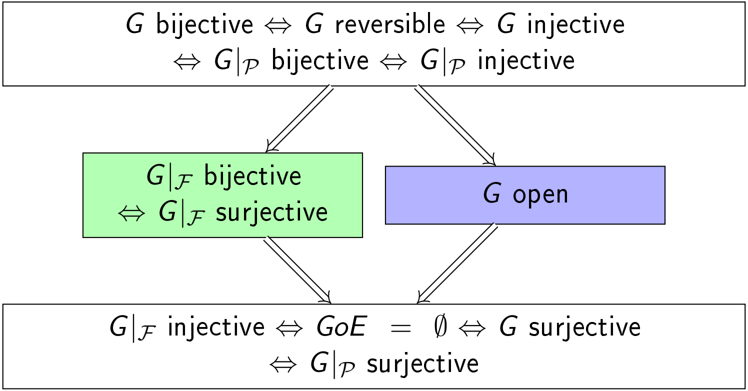
Results

Minimal representants of open rules for ECA

15 51 60 90 105

Openess and surjectivity on finite configurations are distinct properties without implications between them.

Implication diagram



Conclusions

- New decision algorithm for surjectivity on $\mathcal{F}$ for 1D CA
- Completion of the implication diagram of [durand97] in 1D *w.r.t.* the openess property

Open question [kari2013]

There exists a binary CA with neighborhood size between 5 and 10 which is surjective but not closing in either direction? 253.678.110

Futur works

- Study the notion of openess in dimension 2 for some peculiar cases
- Reduce the complexity of the restriction over $\mathcal{F}$

Thank you for your attention :-)

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