Effect of Graph Structure on the Limit Sets of Threshold Dynamical Systems

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with

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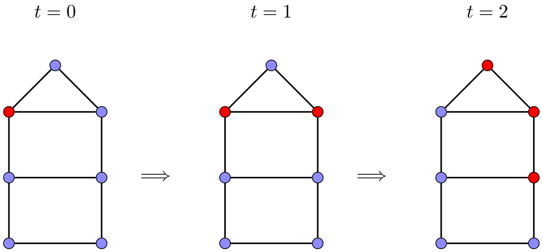
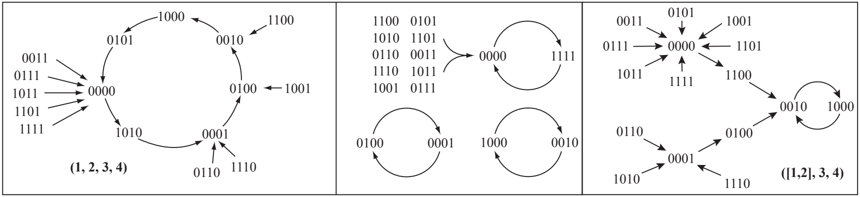
Graph dynamical systems/Automata Networks/Finite dynamical systems



- Constituents:
  - A network
  - Vertices with associated states (Boolean case: 0 or 1)
  - A vertex function of the states of self + neighbors govern local state evolution
  - An update scheme determines the manner in which vertex states are updated

Phase space & limit sets (of deterministic systems)

- Phase space:

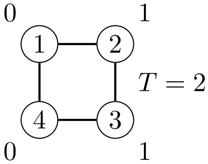


- Long-term behavior and system characteristics:
  - Attractors/limit cycles
  - Fixed points
  - Transient length

Threshold systems

- Let T_v be the threshold associated with vertex v.
- Let $x_v \in \{0,1\}$ be the state of vertex v.
- Let $n[v]$ be the closed neighborhood of v (i.e., including v).

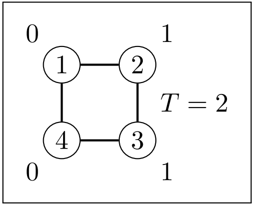
$$f_v\big(n[v]\big) = \begin{cases} 1, & \sum_{w\in n[v]} x_w \geq T_v, \\ 0, & \text{otherwise.} \end{cases}$$

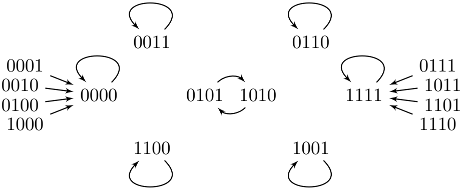


$f_1(x_1, x_2, x_4) = 0$ while,
$f_2(x_1, x_2, x_3) = 1$.

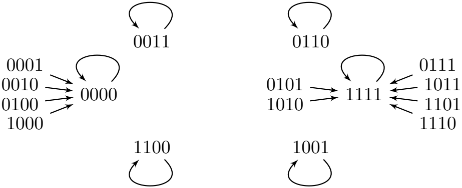
Update schemes

Parallel: all vertices are updated simultaneously. For the 4-cycle network ...





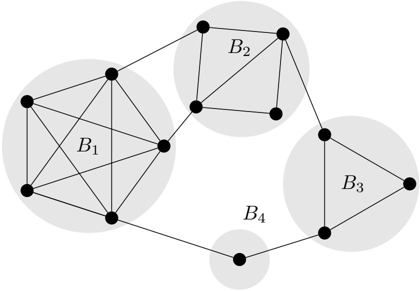
Sequential: vertices updated in a specified order. Here: $(1, 2, 3, 4)$.



Update schemes (contd.)

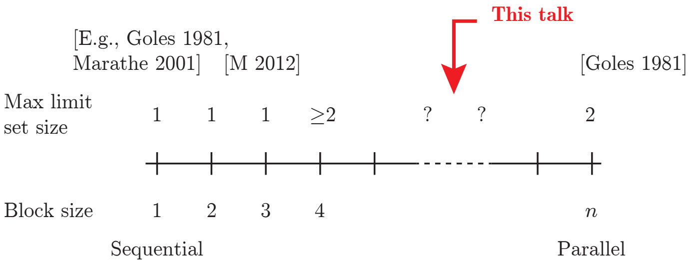
Block-sequential systems:

- The vertex set is partitioned into blocks $B_1, B_2, \ldots, B_k$.
- The blocks are updated sequentially.
- Vertices within blocks are updated synchronously.



- If $|B_i| = 1$ for all i we get a *sequential system*
- With only one block we get a *synchronous system*

Limit cycles in block sequential threshold systems



Conjecture

The maximal periodic orbit size of block sequential threshold systems is 2.

Limit cycles in block sequential threshold systems (ctd.)

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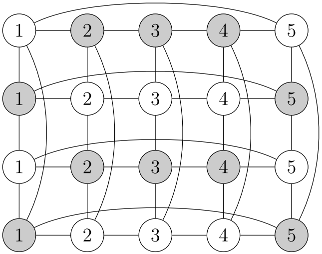


Threshold function: closed majority rule (T_2).
Vertex labels give block ID.

Our results

Can we impose constraints on the network structure so that the block sequential system has only fixed points?

- We provide a sufficient condition based on the potential function method [Barrett et al, 2006].
- We show that several well-known graphs satisfy this condition.
- This extends results of [Mortveit, 2012] where it was shown that systems with block size at most 3 have only fixed points.

The potential function method

- Introduced by [Barrett et al, 2006] to show that sequential threshold systems have only fixed points
- Notation: Recall that T_v is the threshold associated with v, $x_v \in \{0, 1\}$ is its state, and $n[v]$ be the closed neighborhood (i.e., including v).
- Assign potentials to vertices & edges.

Definition (Potential functions)

Vertex potential: $P(x, v) = \begin{cases} T_v, & x_v = 1 \\ \deg(v) - T_v + 2, & x_v = 0\,. \end{cases}$

The edge potential for $e = \{v, v'\}$: $P(x, e) = \begin{cases} 1, & x_v \neq x_{v'} \\ 0, & \text{otherwise.} \end{cases}$

The system potential function for state x is defined as

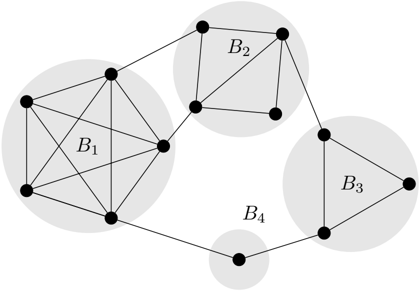
$$P(x) = \sum_{v \in V(X)} P(x, v) + \sum_{e \in E[X]} P(x, e)\,.$$

The potential function method (contd.)

For the sequential threshold systems: [Barrett et al, 2006]

1. The system potential function $P(x) \geq 0$ by definition.
2. The system potential strictly decreases whenever a vertex makes a transition from $0 \longrightarrow 1$ or $1 \longrightarrow 0$.
3. (1) and (2) imply sequential threshold systems have only fixed points as limit cycles.
4. This also implies that the transient length is at most $\left\lfloor \frac{m+n+1}{2} \right\rfloor$.

Block-sequential systems:



The block-sequential map $F : \{0,1\}^n \longrightarrow \{0,1\}^n$ is defined as $F = F_{B_m} \circ F_{B_{m-1}} \circ \cdots \circ F_{B_1}$.

Sufficient condition for block structure

- Given block B, let B' be any induced subgraph. Let x be any assignment of 0s and 1s to vertices of B'.
- Let $\Lambda_{B'}(x)$ be the set of edges in B' with their end points having the same state.
- The phase space has only fixed points if for every B, the following is satisfied:

$$\text{for all } B' \subseteq B : \quad |E[B']| - |V[B']| - 2|\Lambda_v(x)| < 0$$

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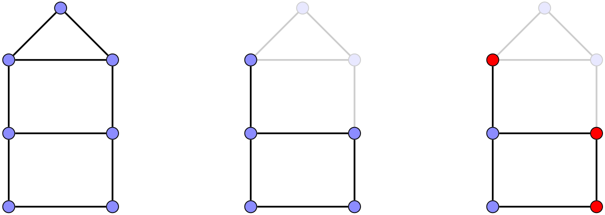

In this example, $|E[B']| = 5$, $|V[B']| = 5$, and $|\Lambda_{B'}(x)| = 2$.

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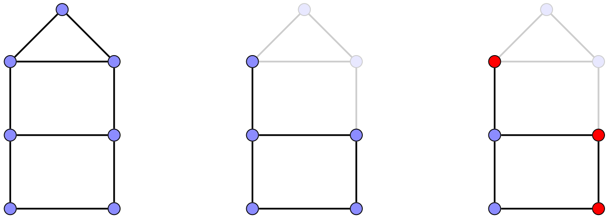

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Note that the condition is independent of the interconnections between the blocks.

Some graph classes which satisfy the condition

1. Trees
2. Odd cycles
3. Complete graph
4. Wheel graph with odd cycle

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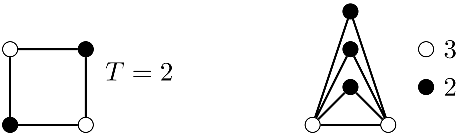
Examples which do not satisfy the condition:



(a) (b)

Examples

Trees

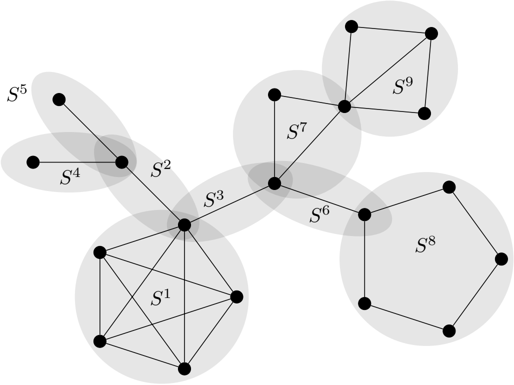
- Every induced subgraph B' is a forest.
- Since every component is independent of one another, we can assume without loss of generality that B' is a tree.
- Then, since $|E[B']| = |V[B']| - 1$, it follows that $|E[B']| - |V[B']| - 2|\Lambda_v(x)| < 0$.

Odd cycles

- Every induced subgraph B' is either a forest or odd cycle (graph itself).
- Forests were covered in the previous example. So let us assume that B' is the odd cycle.
- Then, since $|E[B']| = |V[B']|$, and by pigeon-hole principle, there exists at least one edge with end points in the same state, $|E[B']| - |V[B']| - 2|\Lambda_v(x)| < 0$.

Cut-vertex-free-subgraph decomposition

- Cut-vertex-free-subgraph decomposition corresponds to a(n edge) partition of the graph, where each part induces a maximal cut-vertex free subgraph.
- Suppose a block has a cut-vertex-free-subgraph decomposition such that each subgraph satisfies the sufficient condition, then, the block also satisfies the condition.



Outline of proof for sufficient condition

Let $x' = F_B(x)$, i.e., x' is the configuration obtained from x after updating block B.

Lemma

If B satisfies $|E[B']| - |V[B']| - 2\Lambda_{B'}(y) < 0$, for all induced subgraphs B' and configurations y, then, $P(x') < P(x)$.

Proof.

- Let $B(x, x')$ denote the set of vertices in B such that $x_v \neq x'_v$.
- $P_v(x) = P(x, v) + \sum_{e \in E_v[X]} P(x, e)$. $\Delta P_v = P_v(x') - P_v(x)$ is the change in potential at vertex v.
- It can be shown that $P(x') - P(x) = \sum_{v \in B(x,x')} \Delta P_v$, i.e., the potential difference depends only on the nodes of $B(x, x')$.
- Let γ_v denote the number of neighbors of v in $B(x, x')$ which have the same state as v in x (and therefore, in x'). We show that $\Delta P_v \leq \deg_{B(x,x')}(v) - 2\gamma_v - 2$.

Questions and Acknowledgments

- Is the result tight? Are there more graph classes which satisfy this condition?
- How does block inter-connectivity impact the conclusion?
- Are there better or more interesting results for bithreshold systems?

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