Shrinking One-Way Cellular Automata

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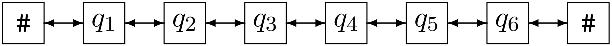
Cellular Automata and Iterative Arrays

A two-way cellular automaton (CA):



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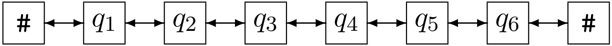

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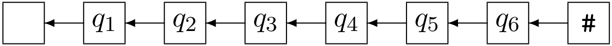

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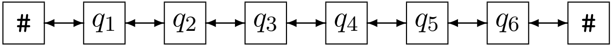

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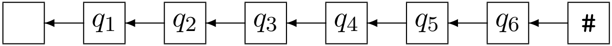
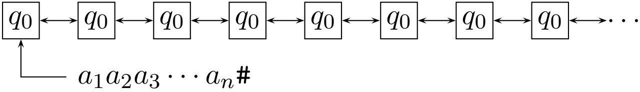
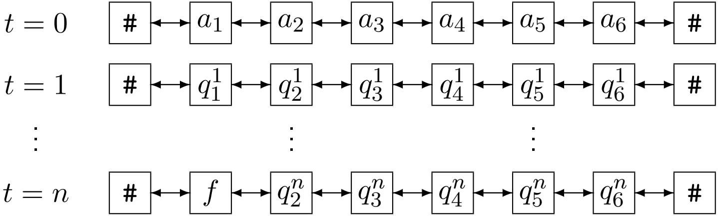


An iterative array (IA) is a cellular automaton with sequential input mode.



Recognizing Formal Languages With Cellular Automata

Input $u = a_1 a_2 \cdots a_6 \in A^+$



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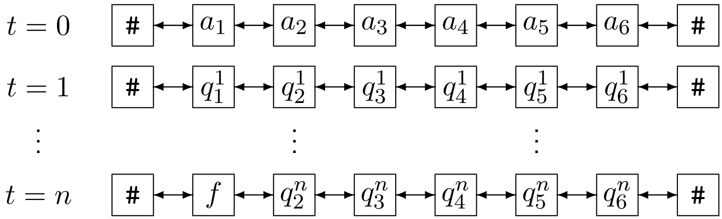

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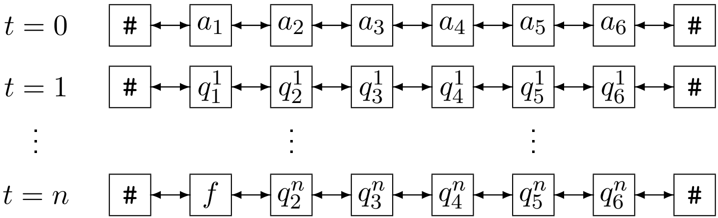

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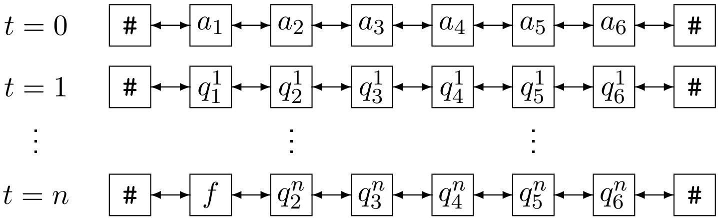

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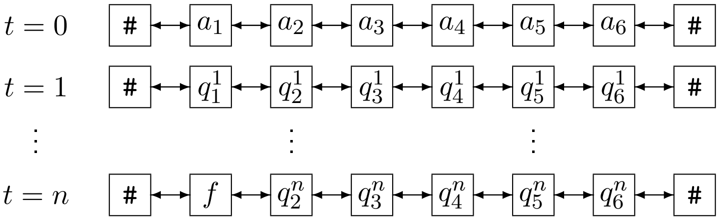

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- ➜ M has time complexity $t : \mathbb{N} \to \mathbb{N}$, $t(n) \geq n$, if all $u \in L(M)$ are accepted within $t(|u|)$ time steps.
- ➜ $\mathscr{L}_t(\mathrm{CA}) = \{ L \mid L \text{ is accepted with time complexity } t \}$

Important Language Classes

- realtime-CA languages $\mathscr{L}_{rt}$(CA) ($t(|u|) = |u|$)
- lineartime-CA languages $\mathscr{L}_{lt}$(CA) ($t(|u|) = m \cdot |u|$, $m \in \mathbb{Q}$, $m \geq 1$)

The language classes for one-way cellular automata $\mathscr{L}_{rt}$(OCA), $\mathscr{L}_{lt}$(OCA) are defined analogously.

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For iterative arrays

- realtime-IA languages $\mathscr{L}_{rt}(\mathsf{IA})$ $(t(|u|) = |u| + 1)$
- lineartime-IA languages $\mathscr{L}_{lt}(\mathsf{IA})$ $(t(|u|) = m \cdot |u|$, $m \in \mathbb{Q}$, $m \geq 1)$

Computational Capacity

$$
\begin{array}{ccccccccc}
 & & \mathrm{DCS} & = & \mathscr{L}(\mathrm{CA}) & = & \mathscr{L}(\mathrm{IA}) & & \\
 & & & & \rotatebox[origin=c]{90}{$\subseteq$} & & & & \\
 & & \mathrm{CF} & \subset & \mathscr{L}(\mathrm{OCA}) & & \rotatebox[origin=c]{90}{$\subseteq$} & & \\
 & & & & \rotatebox[origin=c]{90}{$\subseteq$} & & & & \\
 & & & & \mathscr{L}_{lt}(\mathrm{CA}) & = & \mathscr{L}_{lt}(\mathrm{IA}) & & \\
 & & & & \rotatebox[origin=c]{90}{$\subseteq$} & & \rotatebox[origin=c]{90}{$\subset$} & & \\
 & & \mathrm{DCF} & \subset & \mathscr{L}_{rt}(\mathrm{CA}) & \supset & \mathscr{L}_{rt}(\mathrm{IA}) & \supset & \mathrm{DCF}_{\lambda} \\
 & & & & \parallel & & & & \\
 & & & & \mathscr{L}_{lt}(\mathrm{OCA})^{R} & & & & \\
 & & & & \rotatebox[origin=c]{90}{$\subset$} & & & & \\
\mathrm{REG} & \subset & \mathrm{LCF} & \subset & \mathscr{L}_{rt}(\mathrm{OCA}) & & & &
\end{array}
$$

The language classes $\mathscr{L}_{rt}$(OCA) and $\mathscr{L}_{rt}$(IA) are incomparable.
Both CF and $\mathscr{L}_{rt}$(OCA) and CF and $\mathscr{L}_{rt}$(IA) are incomparable.

Shrinking Cellular Automata – Definition

A shrinking one-way cellular automaton (SOCA) is a system $\langle S, F, A, \#, \delta \rangle$, where

- ➜ S is the finite set of cell states,
- ➜ $F \subseteq S$ is the set of accepting states,
- ➜ $A \subseteq S$ is the finite set of input symbols,
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We define realtime-SOCA languages $\mathscr{L}_{rt}(\text{SOCA})$ with $t(|u|) = |u|$ as usual.

Example

$L = \{\, \$w \mid w \in \{a,b\}^* \text{ and } |w|_a = |w|_b \,\} \in \mathscr{L}_{rt}(\mathsf{SOCA})$.

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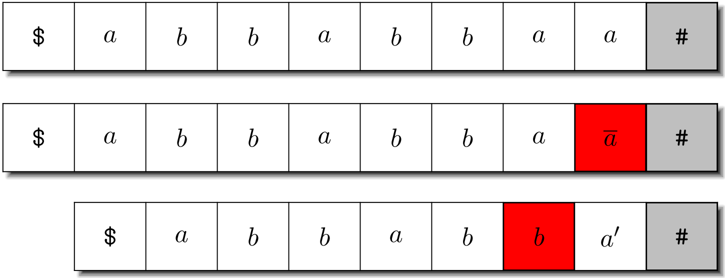

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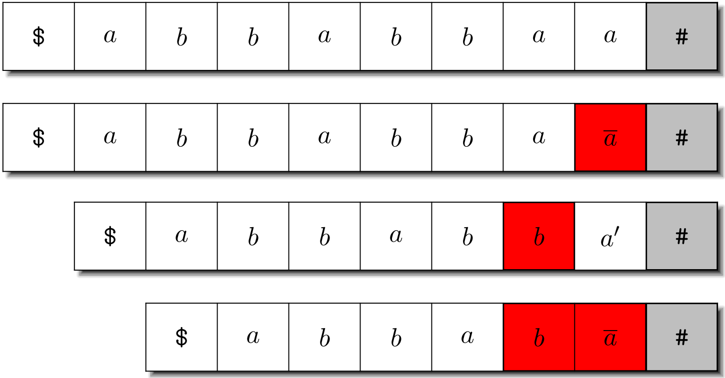

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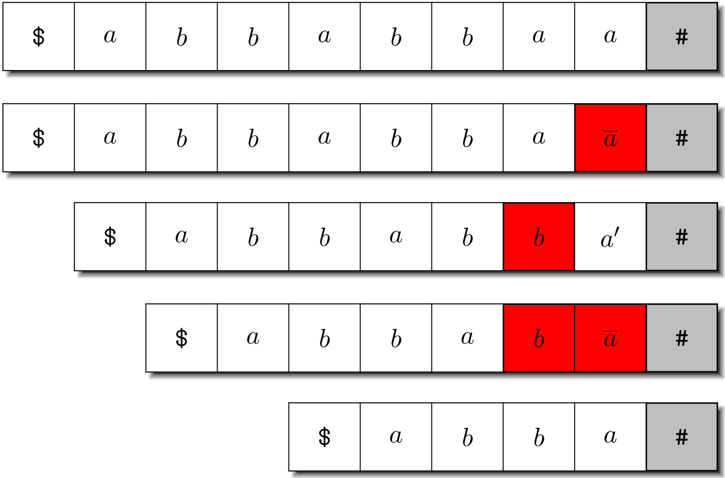

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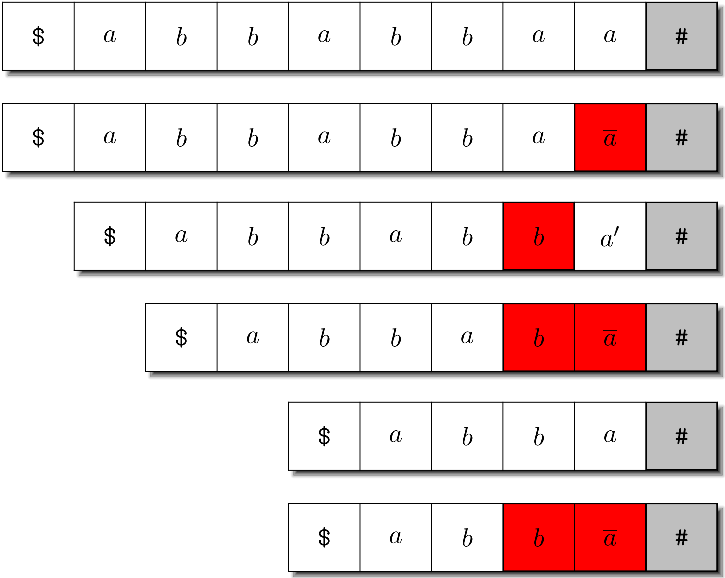

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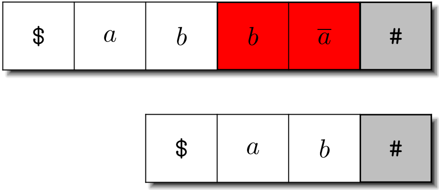

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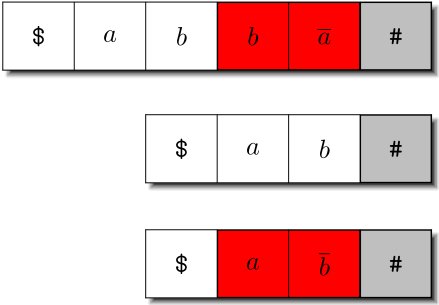

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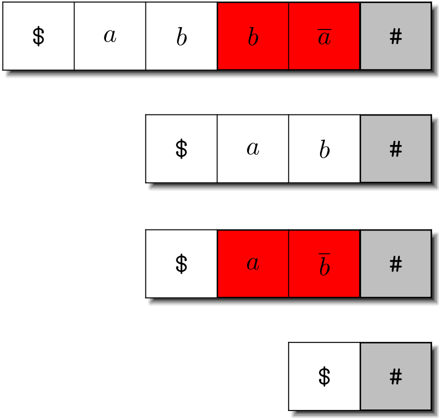

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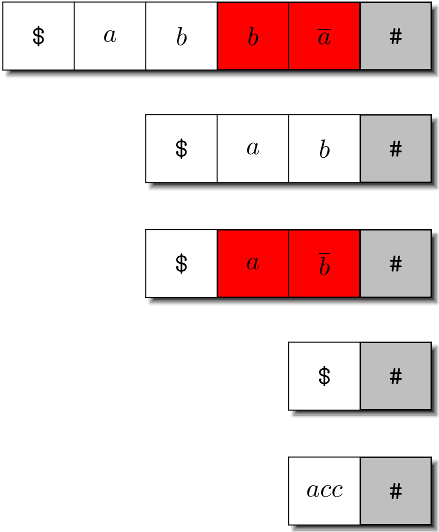

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A valuable tool concerns embeddings. Let $L \subseteq A^*$, \$ $\notin A$, and $\mathrm{emb} : A^* \to A_{\$}^*$, where $\mathrm{emb}(a) = a\$$ for $a \in A$.

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Lemma

Let $r : \mathbb{N} \to \mathbb{N}$ be an increasing function so that $r(O(n)) \leq O(r(n))$. A language L belongs to the family $\mathscr{L}_{n+r(n)}(\mathrm{OCA})$ if and only if $\mathrm{emb}(L)$ belongs to $\mathscr{L}_{n+r(n)}(\mathrm{OCA})$.

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- Let M be an OCA accepting L. An OCA accepting $\mathrm{emb}(L)$ is first sped up suitably. Then, two adjacent cells simulate in two time steps one transition of M.

Computational Capacity (2)

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- An OCA for $\mathrm{emb}(L)$ checks in the first time step the correct format of the input.
- In the second time step, all $-cells are dissolved.
- In the remaining time, L is simulated.

Theorem

Let L be a language from $\mathscr{L}_{lt}(\mathrm{OCA}) \setminus \mathscr{L}_{rt}(\mathrm{OCA})$. Then $\mathrm{emb}(L)$ belongs to $\mathscr{L}_{rt}(\mathrm{SOCA})$ but does not belong to $\mathscr{L}_{rt}(\mathrm{OCA})$. In particular, the family $\mathscr{L}_{rt}(\mathrm{OCA})$ is properly included in $\mathscr{L}_{rt}(\mathrm{SOCA})$.

Real-Time SOCA and Iterative Arrays

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- Remember $\mathrm{DCF}_\lambda \subset \mathscr{L}_{rt}(\mathrm{IA})$.

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Theorem

Let M be an f-SOCA working in real time. Then an equivalent conventional OCA M' working in time $n + f(n)$ can effectively be constructed.

Dissolving versus Time

- Let M be an SOCA and $f : \mathbb{N} \to \mathbb{N}$ be an increasing function.
- If all $w \in L(M)$ are accepted with computations where the number of dissolved cells is bounded by $f(|w|)$, then M is said to be dissolving bounded by f.
- The corresponding class of SOCA is denoted by f-SOCA.

Theorem

Let M be an f-SOCA working in real time. Then an equivalent conventional OCA M' working in time $n + f(n)$ can effectively be constructed.

- Freeze cells instead of dissolving them.
- Slow down the computation.

Dissolving versus Time (2)

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Theorem

Let $r_1, r_2 : \mathbb{N} \to \mathbb{N}$ be two increasing functions. If r_1^{-1} is OCA-constructible, $r_2(O(n)) \leq O(r_2(n))$, and $r_2 \cdot \log(r_2) \in o(r_1)$, then

$$\mathscr{L}_{rt}(r_2\text{-SOCA}) \subset \mathscr{L}_{rt}(r_1\text{-SOCA}).$$

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Steps of the proof:

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Applications:

- $\mathscr{L}_{rt}(n^p\text{-SOCA}) \subset \mathscr{L}_{rt}(n^q\text{-SOCA})$ for two rational numbers $0 \leq p < q \leq 1$.

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- $\mathscr{L}_{rt}(n^p\text{-SOCA}) \subset \mathscr{L}_{rt}(n^q\text{-SOCA})$ for two rational numbers $0 \leq p < q \leq 1$.
- $\mathscr{L}_{rt}(\log^{[j]}\text{-SOCA}) \subset \mathscr{L}_{rt}(\log^{[i]}\text{-SOCA})$ where $\log^{[i]}$ denotes the i-fold iterated logarithms and $0 < i < j$ are integers.