Freezing Cellular Automata

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Definition

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$$F(c)(z) \geqslant c(z) \qquad \forall z \in \mathbb{Z}^d \quad \forall c \in Q^{\mathbb{Z}^d}$$

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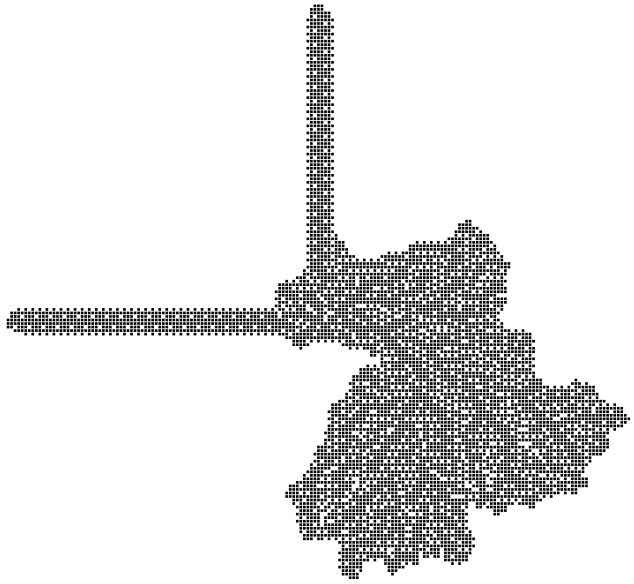
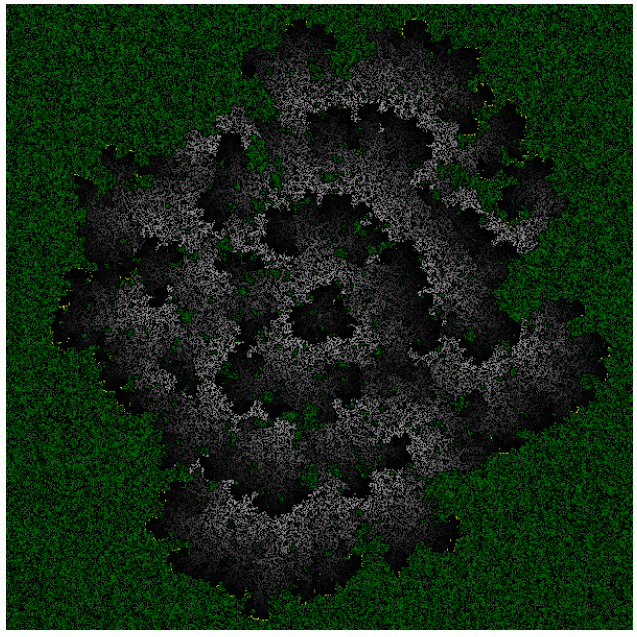
We study **dynamics** and **complexity** of freezing CA.

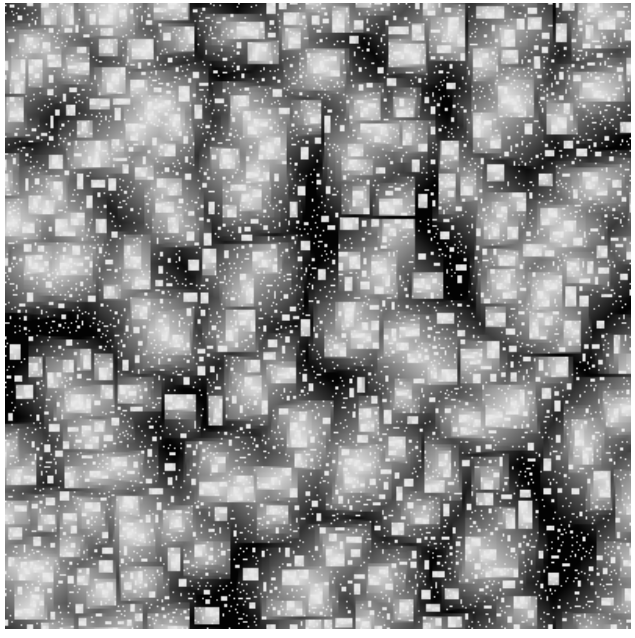
In this talk we focus on **deterministic updates**.



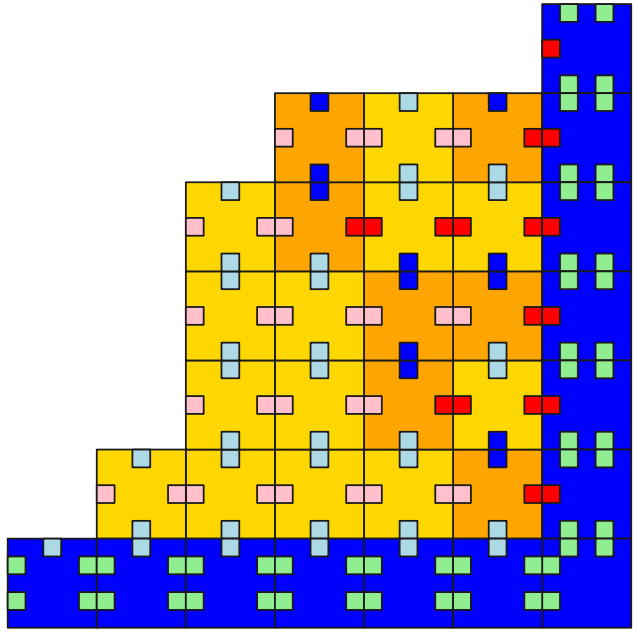
Life without death



Forest fire model



Bootstrap percolation



Self-assembly tiling

Basic dynamical properties

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The only **surjective** FCA is the **identity map**.

Let $f : Q^3 \to Q$ be a FCA over $\geqslant$ different from identity.
Let c be a minimal state such that $f(b, a, d) = c$ with $c > a$.
By minimality if $f(b', a', d') = a$ then $a' = a$.
Thus $|f^{-1}(a)| < |Q|^2$, the CA is unbalanced ♦

Nilpotency

When a FCA is **not nilpotent** then it has **two $\neq$ fixpoints**.

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Nilpotency for FCA is **decidable** in 1D and **undecidable** in higher dimensions.

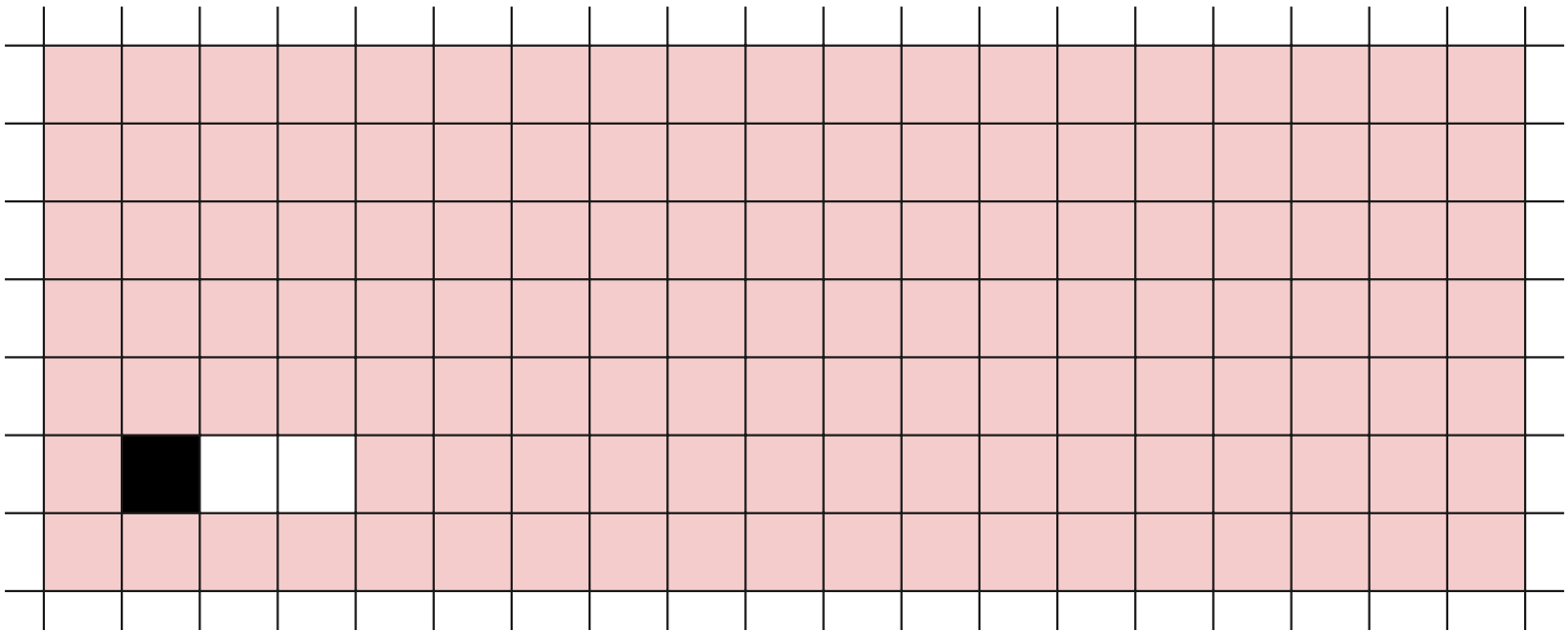
In 1D, detect two fixpoints. In higher dimension, the classical Domino Problem argument is freezing.

Computation complexity in 2D+

Prediction problem PRED$_F$ Given a pattern of states u of size $(2rt+1)^d$ and a state a, determine if $F^t(u) = a$.

There exist a **Turing-universal** 2D FCA with a **P-complete** prediction problem.

Simulate any 1D CA with a 2D FCA with one more state.



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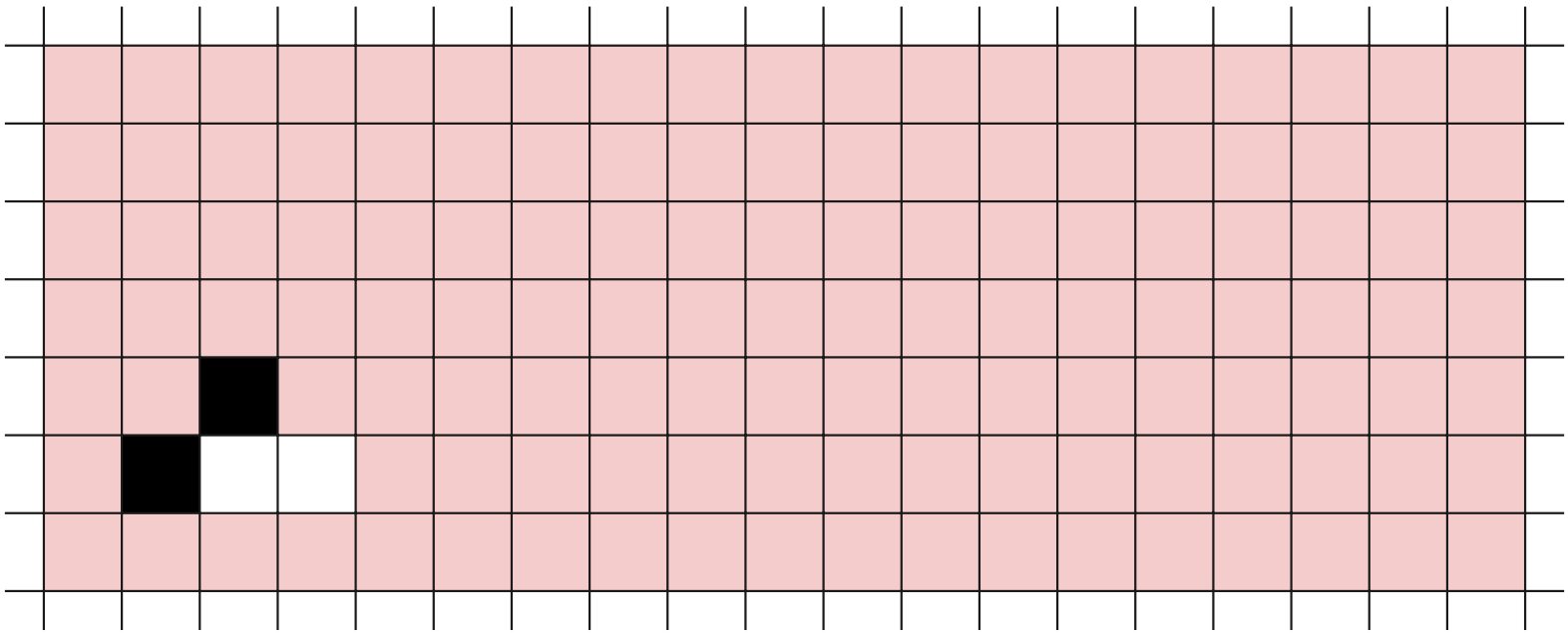

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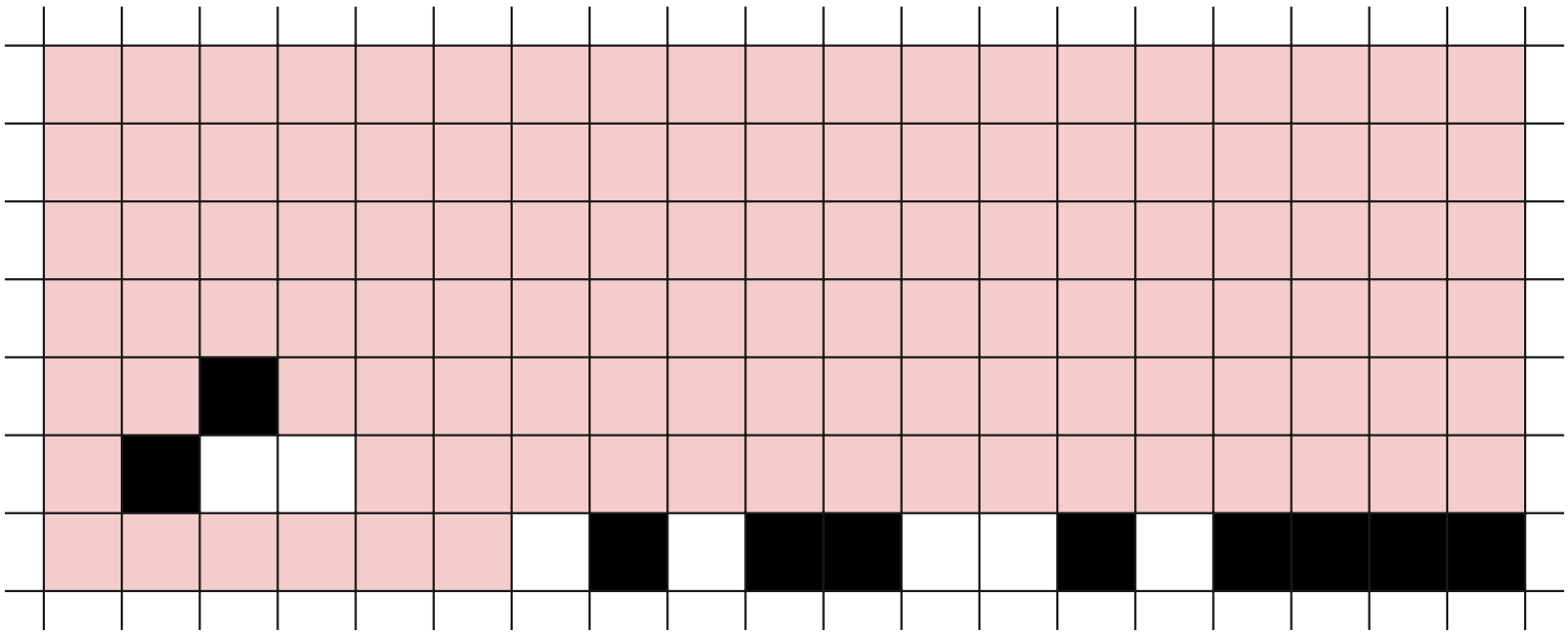

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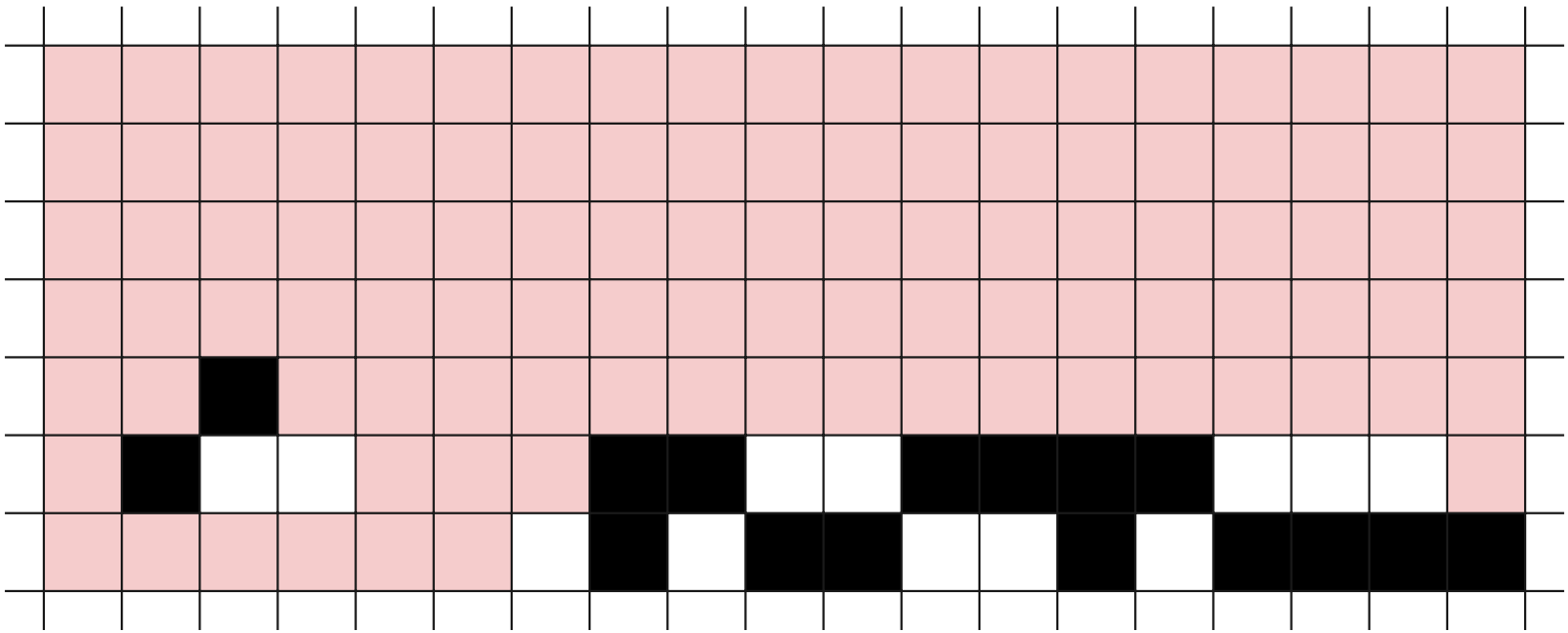

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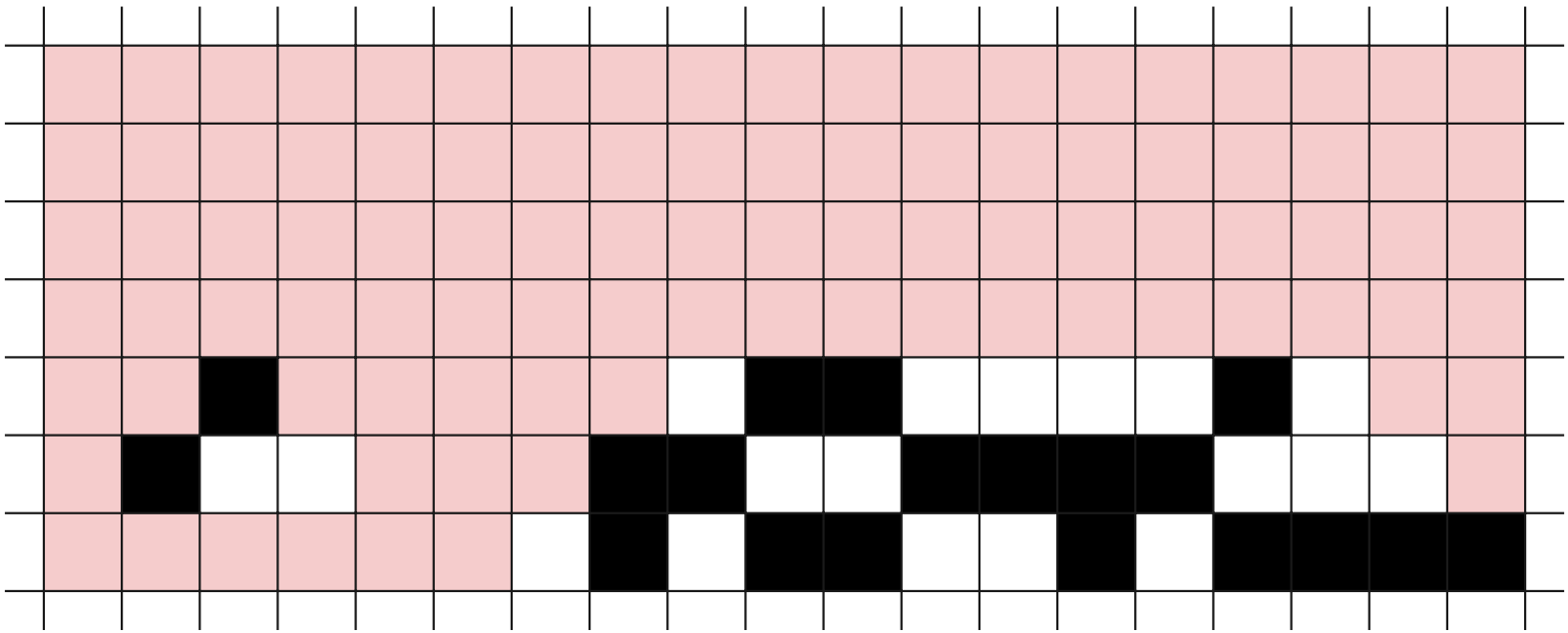

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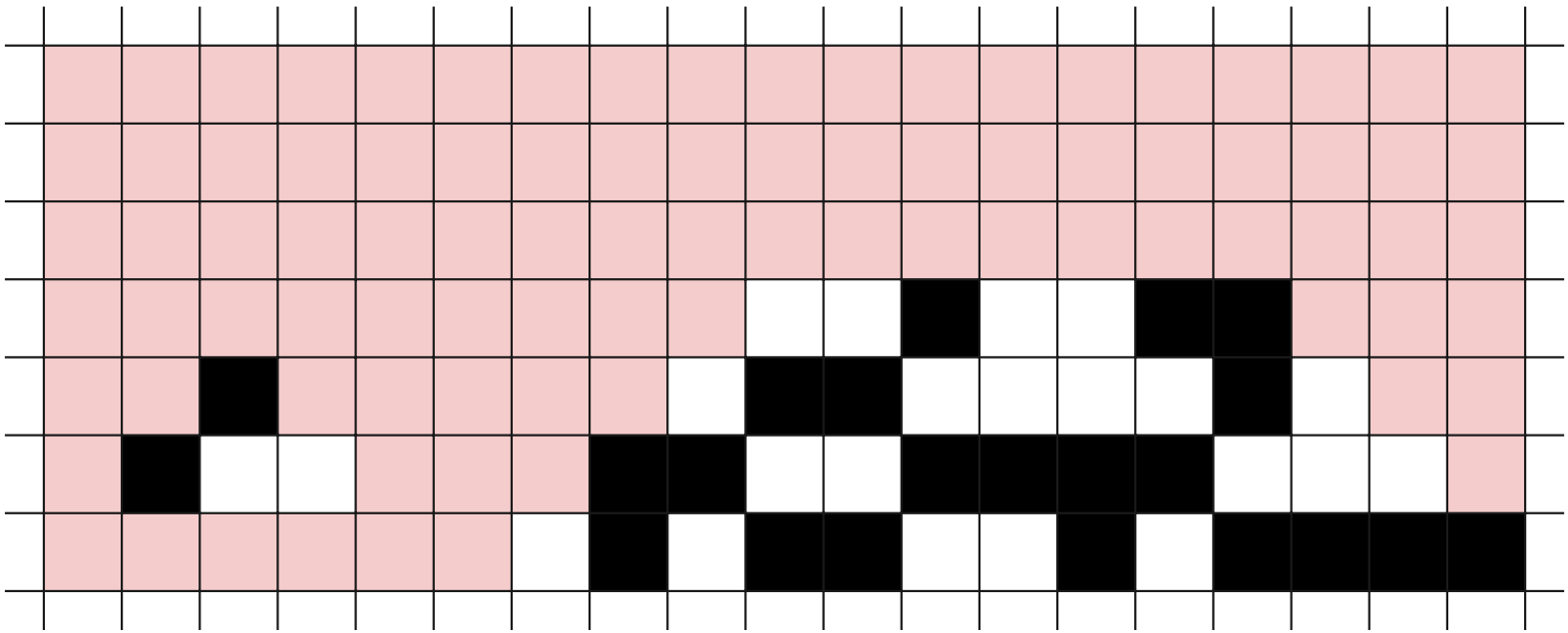

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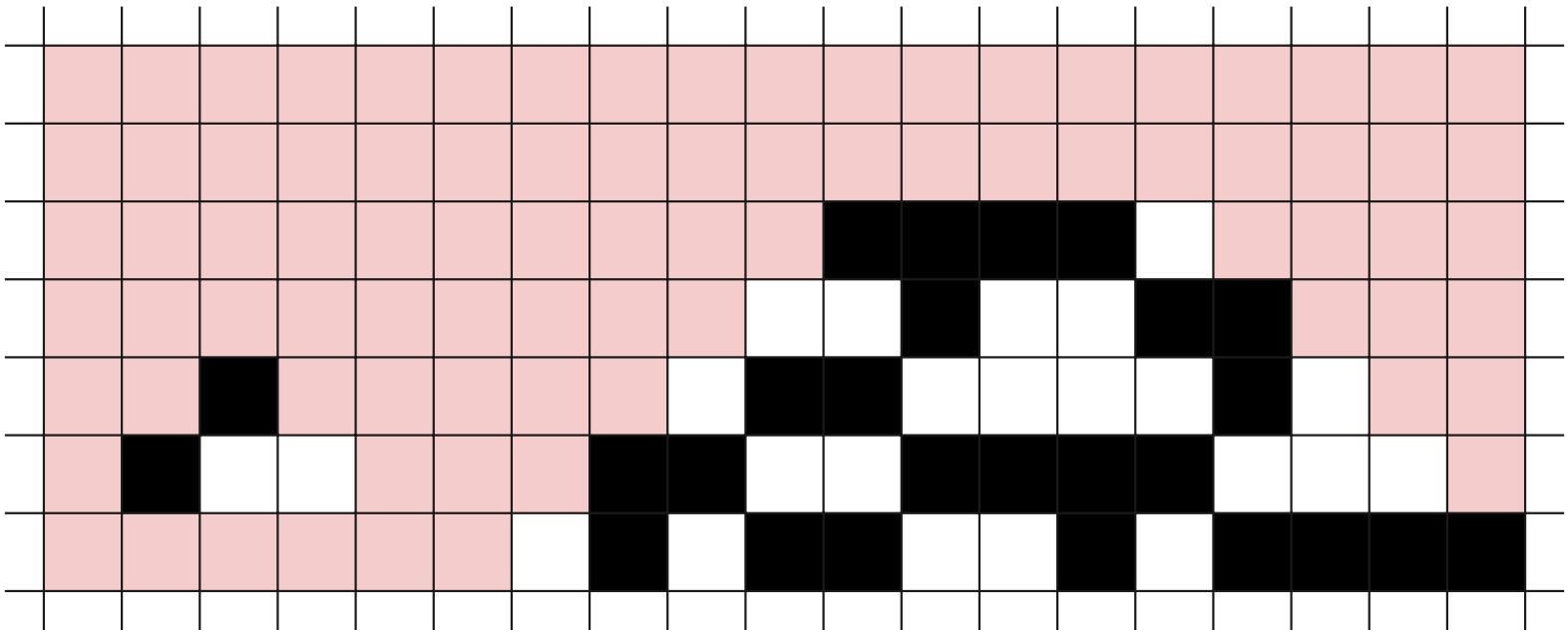

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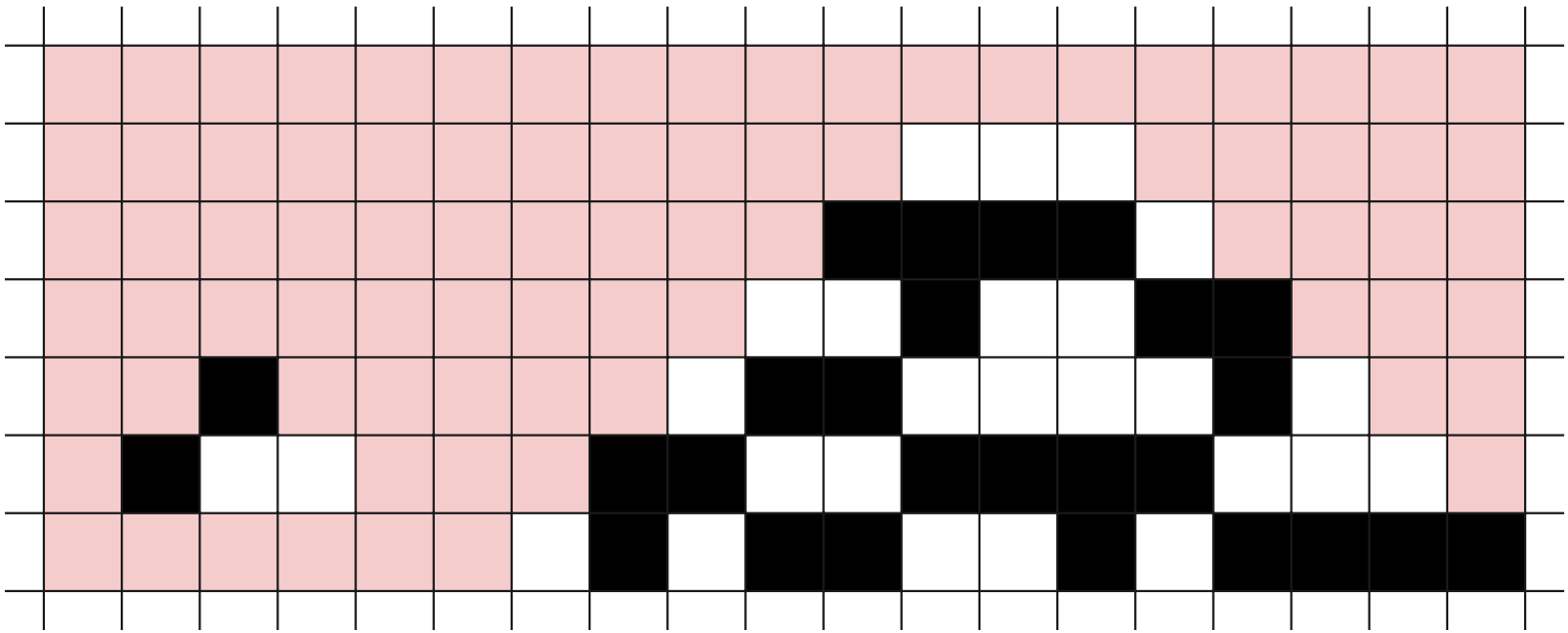

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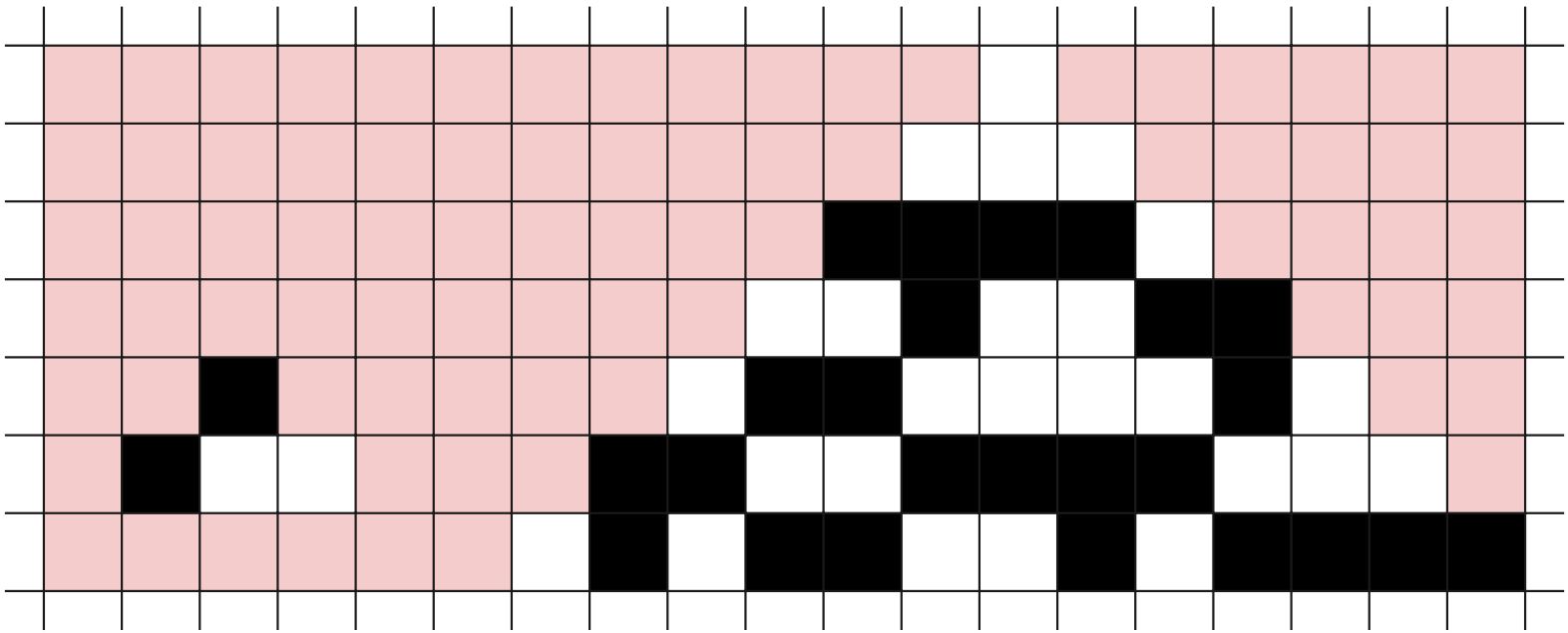

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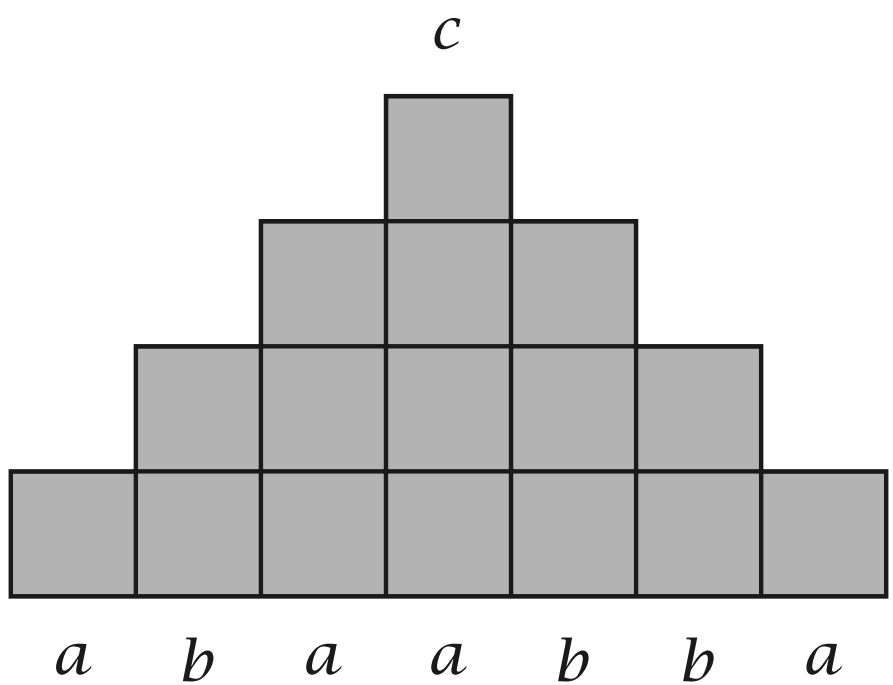

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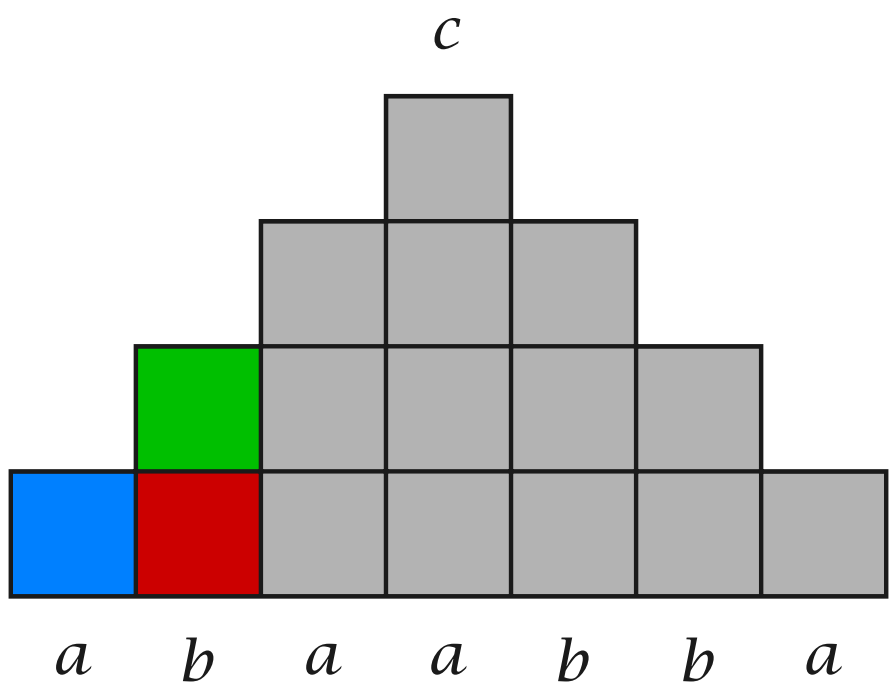

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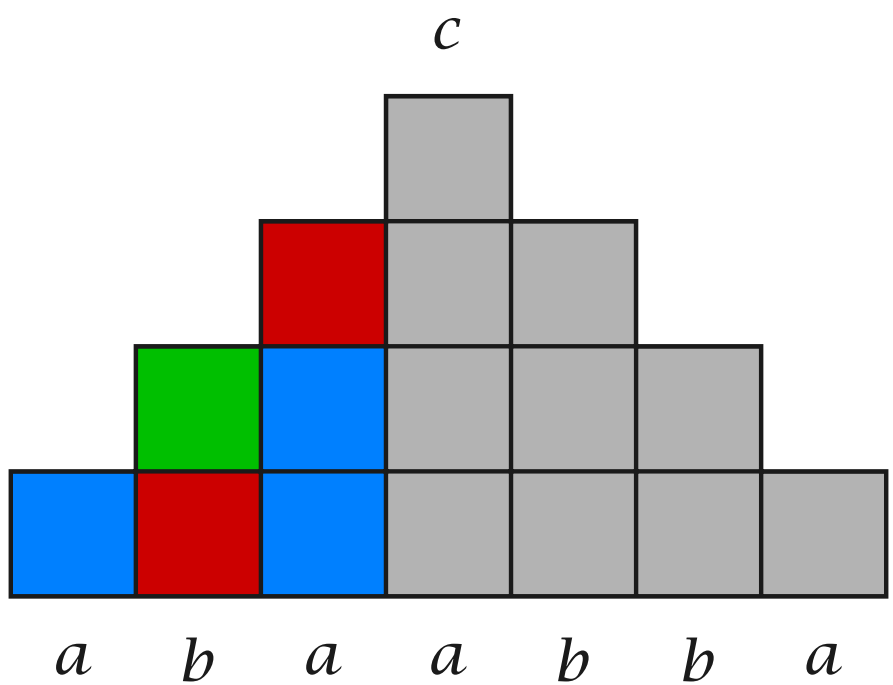

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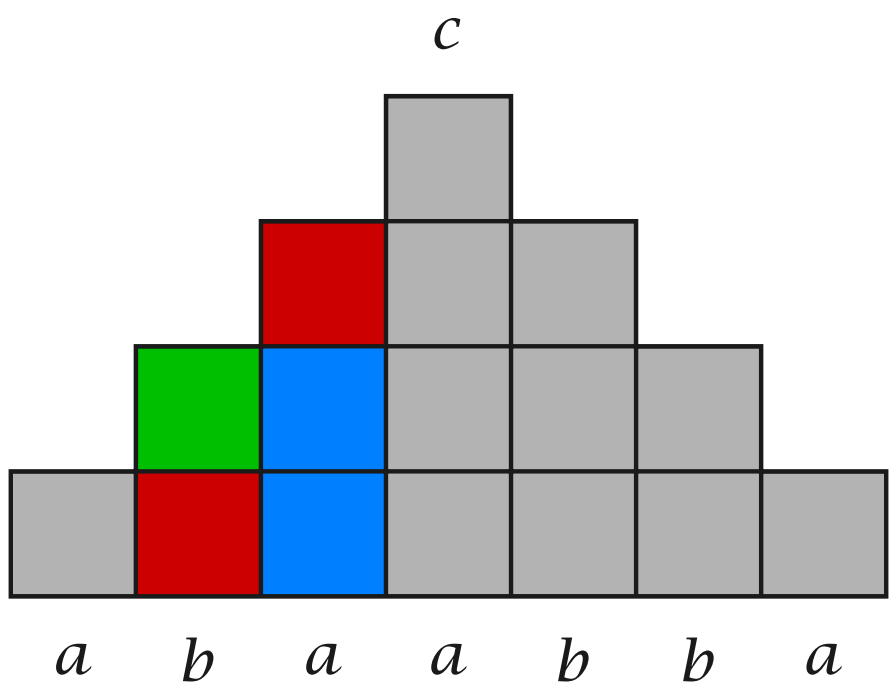

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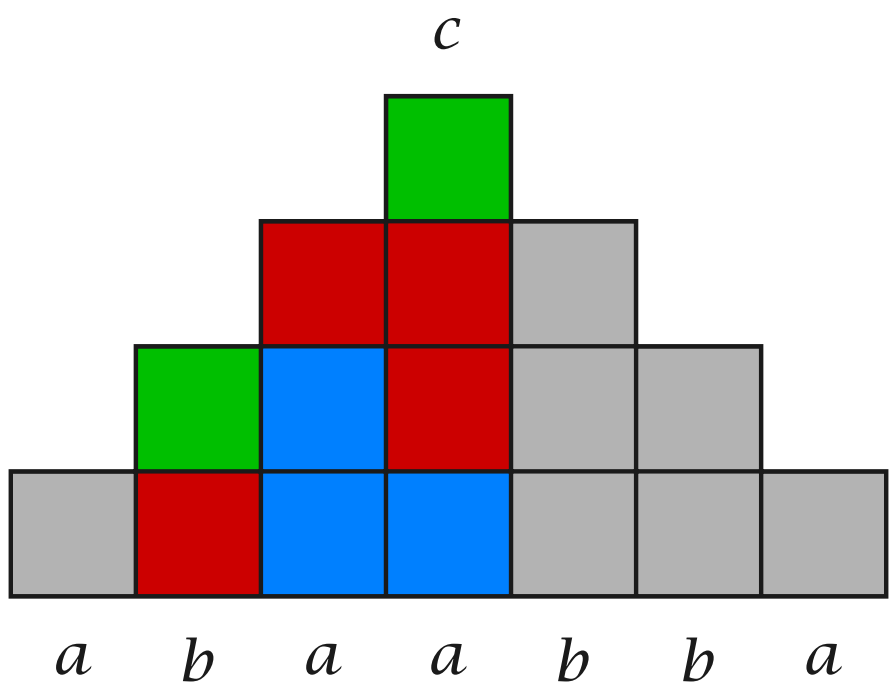

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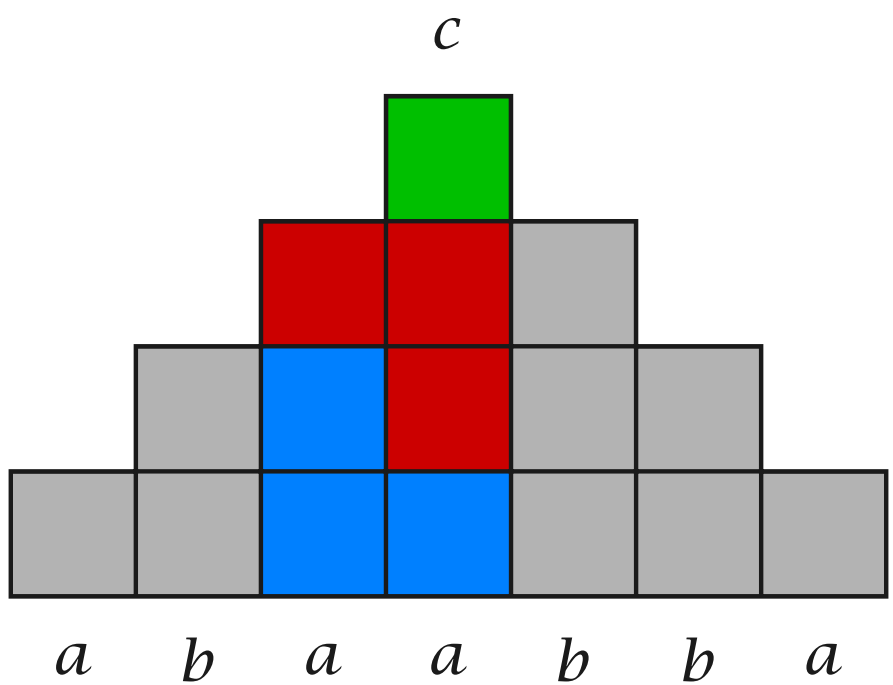

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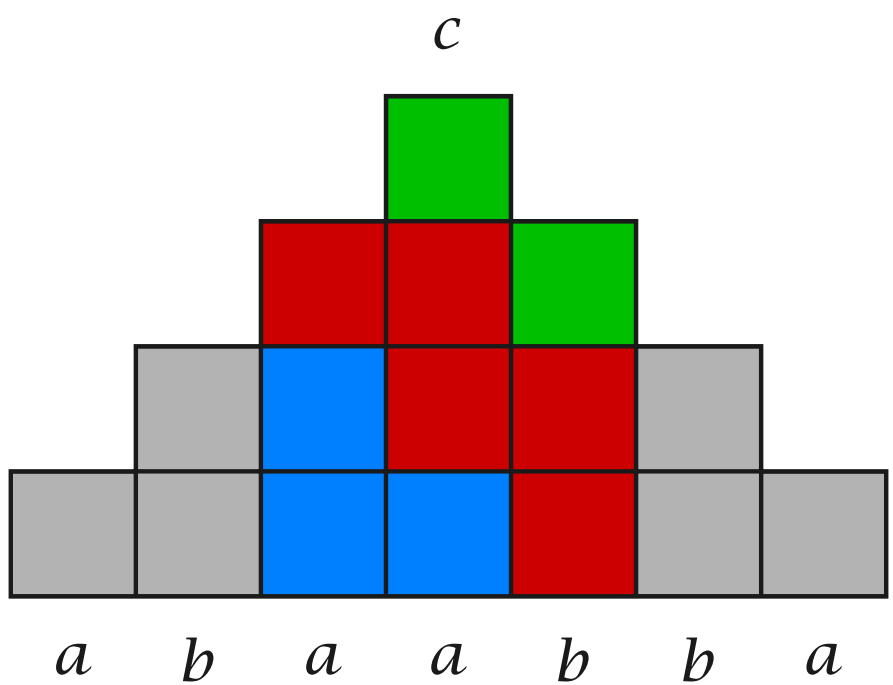

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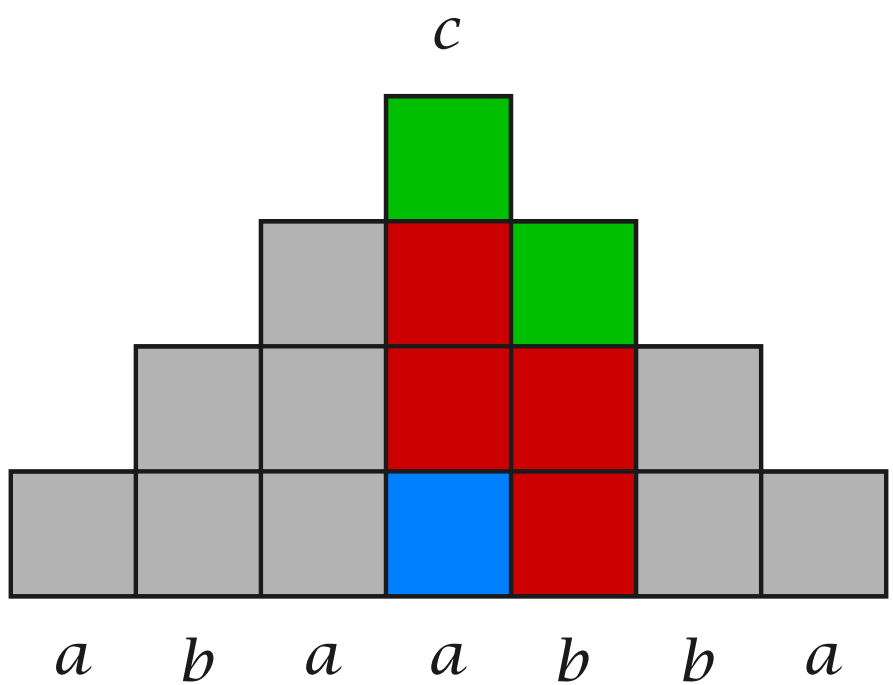

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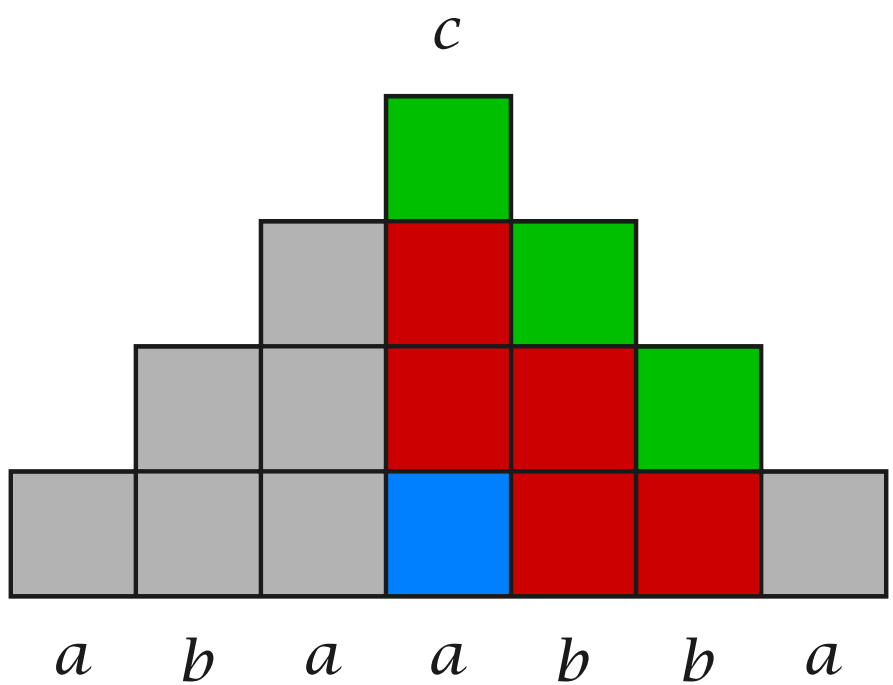

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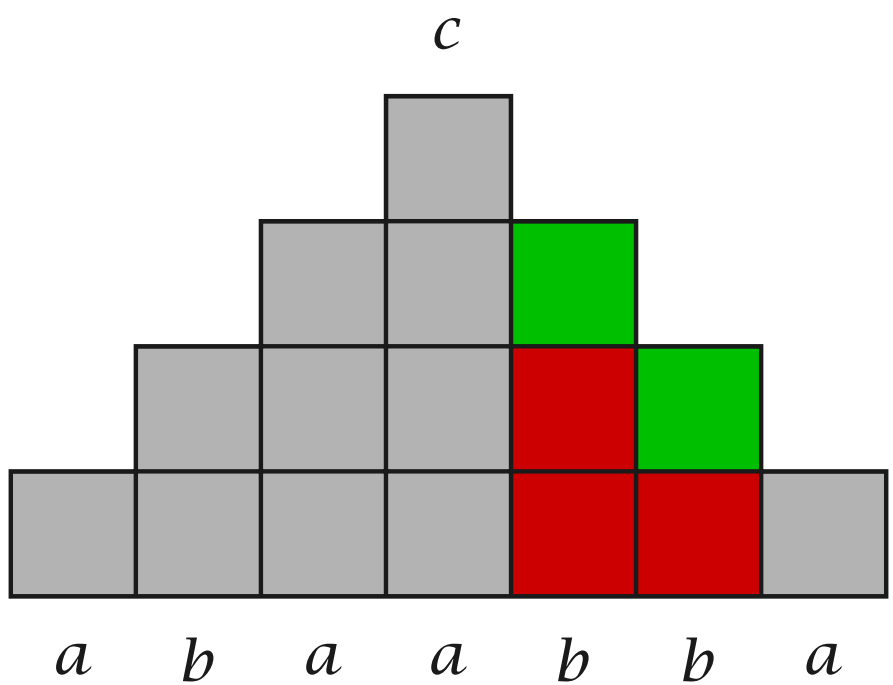

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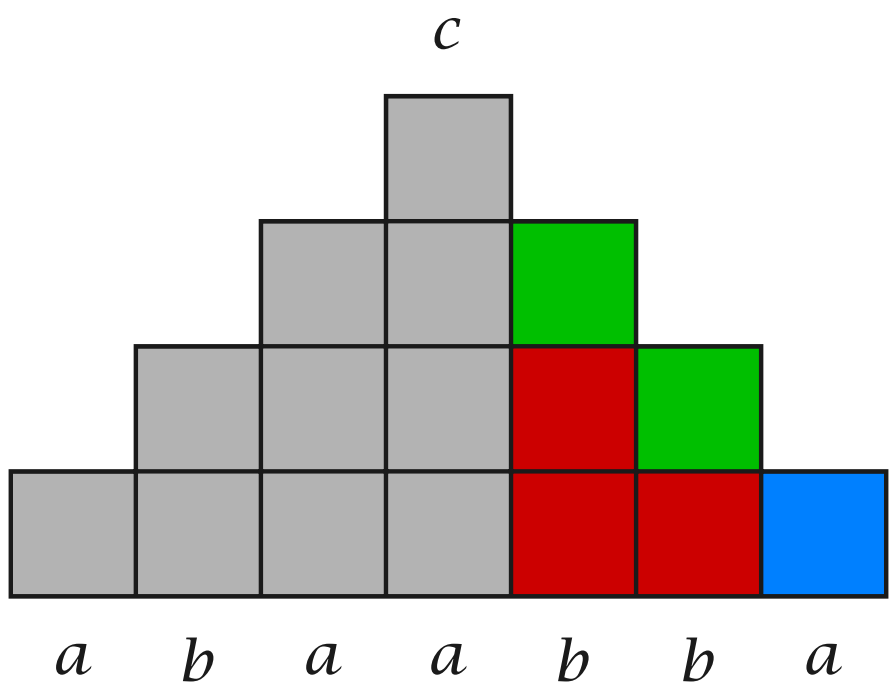

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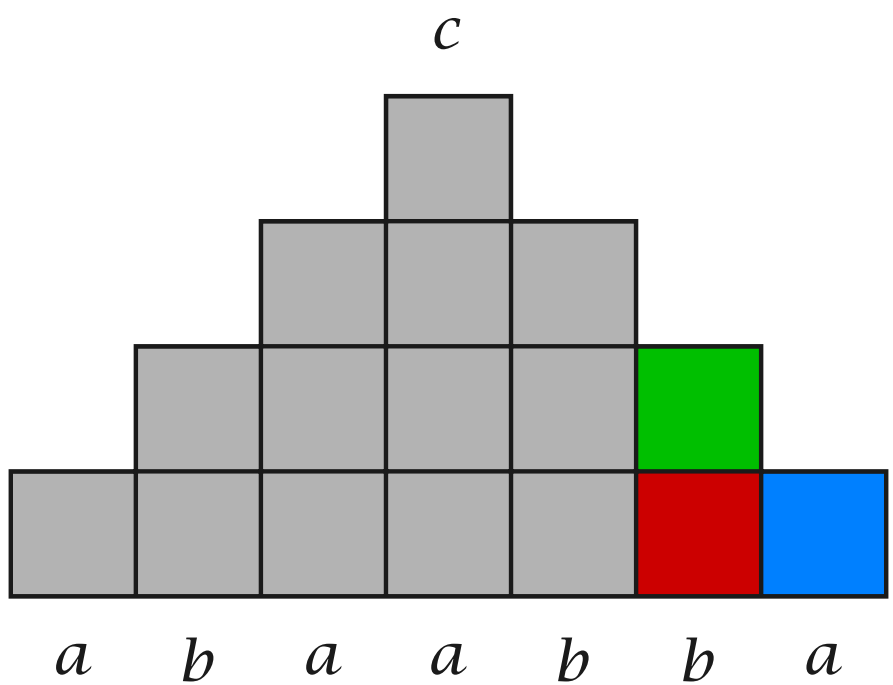

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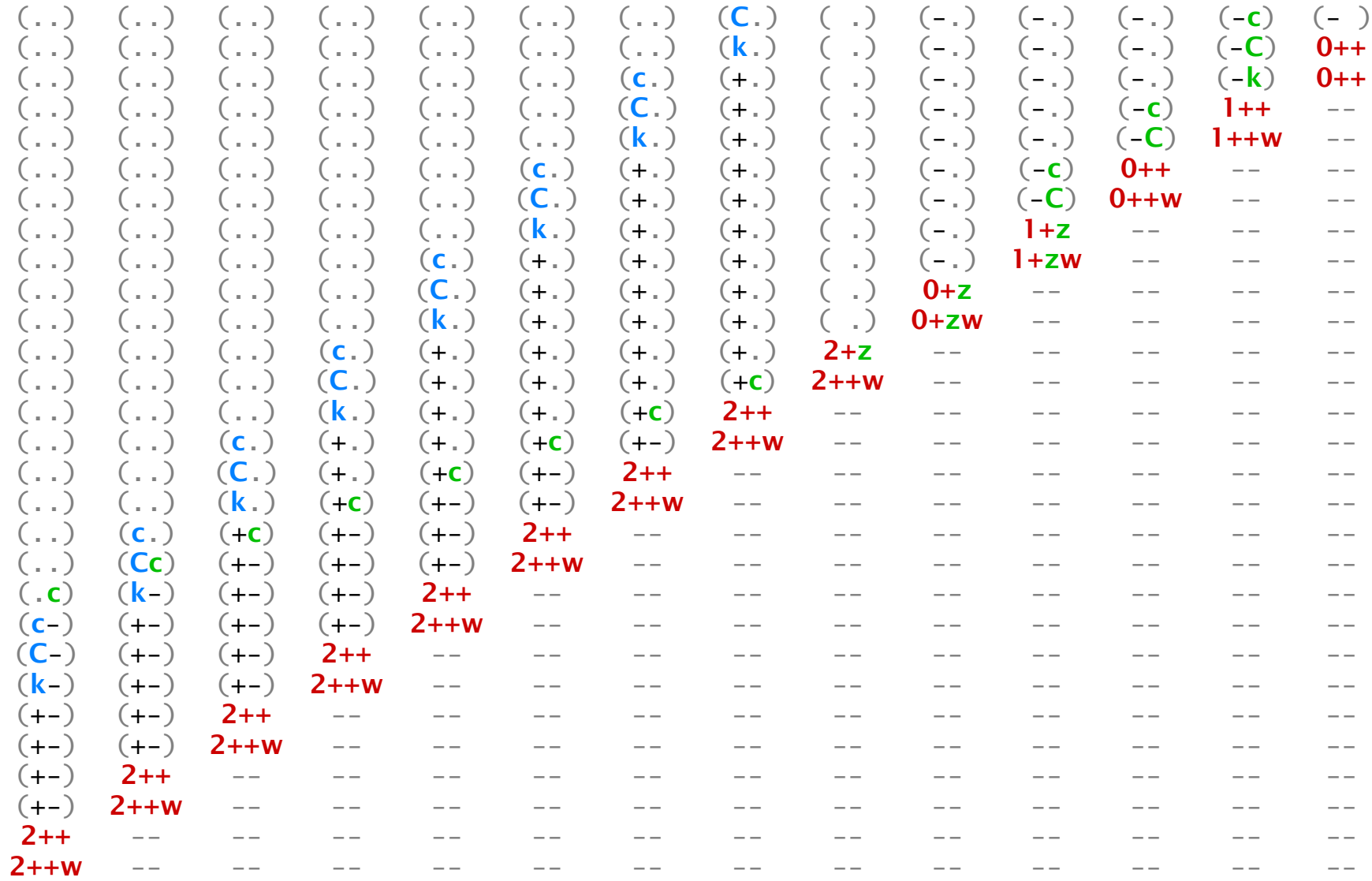
Turing-universality in 1D

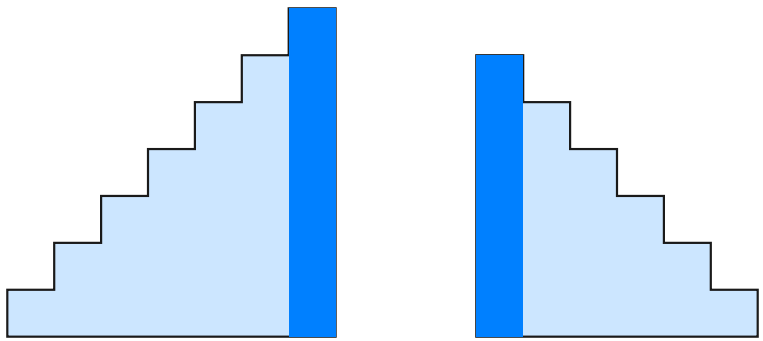
1D FCA are **Turing-universal**.



Communication complexity

Communication complexity of PRED_F is the cost of the best protocol to compute $F^t(u)$ when Alice and Bob both have a part of the pattern u.

Communication complexity of PRED_F is $O(n^{d-1}\log(n))$ on inputs of size n^d for any FCA.



The communicated boundary can be described with $k \times \log(n)$ bits.

What's next

Classify **limit sets** and **dynamics**

Study **intrinsic** freezing universality

Study **asynchronous** update modes

Study particular **order relations**