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Geometric properties of hyperbolic geodesics

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Abstract. In the unit disk $\mathbb{D}$ hyperbolic geodesic rays emanating from the origin and hyperbolic disks centered at the origin exhibit simple geometric properties. The goal is to determine whether analogs of these geometric properties remain valid for hyperbolic geodesic rays and hyperbolic disks in a simply connected region Ω. According to whether the simply connected region Ω is a subset of the unit disk $\mathbb{D}$, the complex plane $\mathbb{C}$ or the extended complex plane (Riemann sphere) $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, the geometric properties are measured relative to the background geometry on Ω inherited as a subset of one of these classical geometries, hyperbolic, Euclidean and spherical. In a simply connected hyperbolic region $\Omega \subset \mathbb{C}$ hyperbolic polar coordinates possess global Euclidean properties similar to those of hyperbolic polar coordinates about the origin in the unit disk if and only if the region is Euclidean convex. For example, the Euclidean distance between travelers moving at unit hyperbolic speed along distinct hyperbolic geodesic rays emanating from an arbitrary common initial point is increasing if and only if the region is convex. A simple consequence of this is the fact that the two ends of a hyperbolic geodesic in a convex region cannot be too close. Exact analogs of this Euclidean separating property of hyperbolic geodesic rays hold when Ω lies in either the hyperbolic plane $\mathbb{D}$ or the spherical plane $\mathbb{C}_\infty$.

Keywords. hyperbolic metric, hyperbolic geodesics, hyperbolic disks, Euclidean convexity, hyperbolic convexity, spherical convexity.

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Contents

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1. Introduction

The results in this expository paper are adapted from [16] and [17] and concern geometric properties of hyperbolic geodesics in a simply connected hyperbolic region Ω and, to a lesser extent, geometric properties of hyperbolic disks. These two references contain many results not mentioned here and as well as the details that are not presented in this largely expository article. In particular, proofs not given in this article can be found in these two references. There are three different cases to consider according to whether the region Ω is a subset of the hyperbolic plane $\mathbb{D}$, the Euclidean plane $\mathbb{C}$, or the spherical plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. Two geometries on the region Ω will be considered. First, the intrinsic hyperbolic geometry on Ω and, second, the geometry that Ω inherits as a subset of the hyperbolic, Euclidean or spherical plane.

Here is a rough description of the types of behavior of hyperbolic geodesics that we will consider. Fix a point $w_0 \in \Omega$. For $\theta \in \mathbb{R}$, let $\rho(w_0, \Omega)$ denote the hyperbolic geodesic ray emanating from w_0 that has unit Euclidean tangent $e^{i\theta}$ at w_0 and let $w_0(s, \theta)$ be the hyperbolic arc length parametrization of this geodesic. Under what conditions does the point $w_0(s, \theta)$ move monotonically away from w_0 when s increases? Here motion away from w_0 is measured relative to the background distance. For example, if Ω lies in the Euclidean plane, this means the Euclidean distance $|w_0(s, \theta) - w_0|$ should increase with s. The second type of behavior we consider is whether the background distance between distinct geodesic rays increases as points move along these rays. In the Euclidean case we inquire whether the Euclidean distance $|w_0(s, \theta_1) - w_0(s, \theta_2)|$ increases with s when $e^{i\theta_1} \neq e^{i\theta_2}$. Intuitively, one can think of two travelers departing from w_0 at the same time along different hyperbolic geodesic rays and traveling at unit hyperbolic speed along the geodesics and asking whether the travelers separate monotonically in the Euclidean sense. Finally, we investigate the shape of hyperbolic circles relative to the background geometry. The main concern is whether hyperbolic circles are convex curves relative to the background geometry. Hyperbolic rays emanating from a point w_0 together with hyperbolic circles centered at w_0 form the coordinate grid for hyperbolic polar coordinates in Ω, so our work can be interpreted as studying geometric properties of the hyperbolic polar coordinate grid relative to the background geometry.

A descriptive outline of the paper follows. Hyperbolic polar coordinates in the unit disk are defined in Section 2, while Section 3 extends hyperbolic polar coordinates to any Euclidean disk or half-plane. Simple Euclidean properties of the hyperbolic polar coordinate grid in any disk or half-plane are established as the model for future investigations. Hyperbolic polar coordinates for a simply connected region are introduced in Section 4. Loosely speaking hyperbolic polar coordinates can be transferred from the unit disk to a simply connected region Ω by using the Riemann Mapping Theorem; a conformal map $f : \mathbb{D} \to \Omega$ is a hyperbolic isometry. Characterizations of Euclidean convex univalent functions are discussed in Section 5. in Section 6 these characterizations are used to establish Euclidean properties of hyperbolic polar coordinates in Euclidean convex regions and to show that these properties characterize Euclidean convex regions. The remainder of the paper is devoted to analogs of these results in the spherical and hyperbolic planes. The spherical plane is introduced in Section 7 along with the notion of a spherically convex region. The results for regions in the spherical plane parallels the Euclidean context. Spherically convex univalent functions are presented in Section 8. The reader should note the number of parallels between spherically convex univalent functions and Euclidean convex univalent functions. The results for spherically convex univalent functions seem more involved than those for Euclidean convex univalent functions; the more complicated nature of formulas relating to spherically convex univalent functions is due to the fact that the spherical metric has curvature 1 while the Euclidean metric has curvature 0. Nonzero curvature causes the appearance of extra terms. Applications of some results for spherically convex functions to the behavior of the hyperbolic coordinate grid in a spherically convex region are given in Section 9. Section 10 considers the behavior of the hyperbolic polar coordinate grid for hyperbolically convex regions in the unit disk. Because of the strong similarity with the previous cases for Euclidean convexity and spherical convexity, we present a concise discussion of the results. The reader should note that some theorems for hyperbolically convex univalent functions formally differ from those for spherically convex univalent functions by certain sign changes; these alterations in sign are due to the fact that the hyperbolic plane has curvature -1 while the spherical plane has curvature 1. The brief final section directs the reader to some other situations in function theory in which there are parallel results for the hyperbolic, Euclidean and spherical planes.

2. Hyperbolic polar coordinates in the unit disk

We begin by recalling the unit disk as a model of the hyperbolic plane. The hyperbolic metric on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ is

$$\lambda_{\mathbb{D}}(z)|dz| = \frac{2|dz|}{1 - |z|^2}.$$

The hyperbolic metric has curvature -1; that is,

$$-\frac{\triangle \log \lambda_{\mathbb{D}}(z)}{\lambda_{\Omega}^2(z)} = -1,$$

where $z = x + iy$ and

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

denotes the usual Laplacian. For any piecewise smooth curve γ in $\mathbb{D}$ the hyperbolic length of γ is given by

$$\ell_{\mathbb{D}}(\gamma) = \int_{\gamma} \lambda_{\mathbb{D}}(z)|dz|.$$

The hyperbolic distance between $z, w \in \mathbb{D}$ is defined by

$$d_{\mathbb{D}}(z,w) = \inf \ell_{\mathbb{D}}(\gamma),$$

where the infimum is taken over all piecewise smooth paths γ in $\mathbb{D}$ that join z and w. In fact,

$$d_{\mathbb{D}}(a,b) = 2\tanh^{-1}\left|\frac{a-b}{1-\bar{b}a}\right|.$$

The group $\mathcal{A}(\mathbb{D})$ of conformal automorphisms of the unit disk is the set of holomorphic isometries of the hyperbolic metric and also of the hyperbolic distance. A path γ joining z to w is called a hyperbolic geodesic arc if $d_{\mathbb{D}}(z,w) = \ell_{\mathbb{D}}(\gamma)$. The *(hyperbolic) geodesic* through z and w is $C \cap \mathbb{D}$, where C is the unique Euclidean circle (or straight line) that passes through z and w and is orthogonal to the unit circle $\partial\mathbb{D}$. If γ is any piecewise smooth curve joining z to w in $\mathbb{D}$, then the hyperbolic length of γ is $d_{\mathbb{D}}(z,w)$ if and only if γ is the arc of C in $\mathbb{D}$ that joins z and w. A hyperbolic disk in the unit disk is $D_{\mathbb{D}}(a,r) = \{z : d_{\mathbb{D}}(a,z) < r\}$, where $a \in \mathbb{D}$ is the hyperbolic center and $r > 0$ is the hyperbolic radius. A hyperbolic disk in $\mathbb{D}$ is Euclidean disk with closure contained in $\mathbb{D}$. In fact, $D_{\mathbb{D}}(a,r)$ is the Euclidean disk with center c and radius R, where

$$c = \frac{a\left(1-\tanh^2(r/2)\right)}{1-|a|^2\tanh^2(r/2)} \quad \text{and} \quad R = \frac{(1-|a|^2)\tanh(r/2)}{1-|a|^2\tanh^2(r/2)}.$$

For more details about hyperbolic geometry on the unit disk the reader should consult [1].

Hyperbolic polar coordinates on the unit disk relative to a specified pole or center are defined as follows. Fix a point a in $\mathbb{D}$, called the *pole* or *center* for polar coordinates based at a. For θ in $\mathbb{R}$ let $\rho_\theta(a,\mathbb{D}) = \rho_\theta(a)$ denote the unique hyperbolic geodesic ray emanating from a that is tangent to the Euclidean unit vector $e^{i\theta}$ at a. For $\theta = 0$ the Euclidean unit tangent vector is 1 and $\rho_0(a)$ is called the *horizontal* hyperbolic geodesic emanating from a because the unit tangent vector at a is horizontal. Of course, $\rho_{\theta+2n\pi}(a) = \rho_\theta(a)$ for all n in $\mathbb{Z}$. Let $s \mapsto z_a(s,\theta)$, $0 \le s < +\infty$, be the hyperbolic arc length parametrization of $\rho_\theta(a)$. This means

$$\frac{\partial z_a(s,\theta)}{\partial s} = \frac{e^{i\Theta(s,\theta)}}{\lambda_{\mathbb{D}}(z_a(s,\theta))}, \tag{2.1}$$

where $e^{i\Theta(s,\theta)}$ is a Euclidean unit tangent to $\rho_\theta(a)$ at the point $z_a(s,\theta)$. For fixed θ the point $z_a(s,\theta)$ moves along the geodesic ray $\rho_\theta(a)$ with unit hyperbolic speed. Two hyperbolic geodesic rays with distinct unit tangent vectors at a are disjoint except for their common initial point and $\mathbb{D} = \cup\{\rho_\theta : 0 \le \theta < 2\pi\}$. For each z in $\mathbb{D} \setminus \{a\}$ there is a unique geodesic ray $\rho_\theta(a)$ with $0 \le \theta < 2\pi$ that contains z, so there exist unique $s > 0$ and θ in $[0, 2\pi)$ with $z_a(s,\theta) = z$. The *hyperbolic polar coordinates* of the point z relative to the center or pole at a are the ordered pair (s,θ), where $z_a(s,\theta) = z$. The first coordinate, $s = d_{\mathbb{D}}(a,z)$, is the hyperbolic distance from a to z and the second polar coordinate, θ, is the angle between the horizontal hyperbolic geodesic ray $\rho_0(a)$ and the ray $\rho_\theta(a)$ that contains z at the pole a. The hyperbolic circle with hyperbolic center a and hyperbolic radius s is $c_{\mathbb{D}}(a,s) = \{z : d_{\mathbb{D}}(a,z) = s\}$. Note that each geodesic ray $\rho_\theta(a)$ is orthogonal to every hyperbolic circle $c_\Omega(a,s)$. Thus, the coordinate grid for hyperbolic polar coordinates based at a consists of hyperbolic geodesics emanating from a and hyperbolic circles centered at a. In terms of hyperbolic polar coordinates

$$\lambda_{\mathbb{D}}^2(z)(dx^2 + dy^2) = ds^2 + \sinh^2(s)d\theta^2.$$

For $a = 0$, $\rho_\theta(0)$ is the radial segment $[0, e^{i\theta})$ with hyperbolic arc length parametrization $z_0(s,\theta) = \tanh(s/2)e^{i\theta}$ and

$$\frac{\partial z_0(s,\theta)}{\partial s} = \frac{e^{i\theta}}{\lambda_{\mathbb{D}}(z_0(s,\theta))} = \frac{1 - |z_0(s,\theta)|^2}{2|z_0(s,\theta)|} z_0(s,\theta). \tag{2.2}$$

Hyperbolic polar coordinates about the origin can be transported to any other center in the unit disk by a hyperbolic isometry. Recall that each conformal automorphism of $\mathbb{D}$ is an isometry of the hyperbolic metric and the hyperbolic distance. For $a \in \mathbb{D}$ the Möbius transformation $f(z) = (z + a)/(1 + \bar{a}z)$ is a conformal automorphism of $\mathbb{D}$ that sends the origin to a and $f'(0) = (1-|a|^2) > 0$. The fact that $f'(0) > 0$ insures that $f(\rho_\theta(0)) = \rho_\theta(a)$ for all $\theta \in \mathbb{R}$ and so $z_a(s,\theta) = f(z_0(s,\theta))$ provides an explicit hyperbolic arc length parametrization of $\rho_a(\theta)$:

$$z_a(s,\theta) = \frac{\tanh(s/2)e^{i\theta} + a}{1 + \bar{a}\tanh(s/2)e^{i\theta}}.$$

Trivially, the Euclidean distance from $a = 0$ to $z_0(s,\theta)$ is an increasing function of s for each fixed θ and the Euclidean distance between $z_0(s,\theta_1)$ and $z_0(s,\theta_2)$ is an increasing function of s when $e^{i\theta_1} \neq e^{i\theta_2}$. It is plausible that these Euclidean properties remain valid for any center $a \in \mathbb{D}$. Rather than investigating these assertions for the special case of the unit disk, we wait to consider the analogous questions in any disk or half-plane. Also, hyperbolic circles centered at the origin are Euclidean circles.

3. Hyperbolic polar coordinates in a disk or half-plane

We let Δ denote any Euclidean disk or half-plane when it is not necessary to distinguish between the cases; otherwise, we use D for a Euclidean disk and H for a Euclidean half-plane. Given Δ there is a Möbius transformation f that maps Δ onto the unit disk. Then the hyperbolic metric on Δ is given by

$$\lambda_\Delta(z) = \lambda_{\mathbb{D}}(f(z))|f'(z)|.$$

This defines the hyperbolic density λ_Δ independent of the Möbius map of Δ onto the unit disk. If $D = \{z : |z-a| < r\}$, then

$$\lambda_D(z)|dz| = \frac{2r|dz|}{r^2 - |z-a|^2}.$$

If H is any half-plane, then

$$\lambda_H(z)|dz| = \frac{|dz|}{d(z, \partial H)},$$

where $d(z, \partial H)$ denotes the Euclidean distance from z to the boundary of H. In particular, for the upper half-plane $\mathbb{H} = \{z : \mathrm{Im}\,(z) > 0\}$,

$$\lambda_{\mathbb{H}}(z)|dz| = \frac{|dz|}{\mathrm{Im}\,(z)}.$$

Because Möbius transformations map circles onto circles, hyperbolic geodesics in a disk or half-plane are arcs of circles orthogonal to the boundary. Also, hyperbolic disks are Euclidean disks with closure contained in the disk or half-plane. Any Möbius map from Δ onto $\mathbb{D}$ is an isometry from Δ with the hyperbolic metric to $\mathbb{D}$ with the hyperbolic metric. See [1] for details.

Hyperbolic polar coordinates are defined on Δ analogous to the definition for the unit disk. Fix a point w_0 in Δ. For θ in $\mathbb{R}$ let $\rho_\theta(w_0, \Delta)$ denote the unique hyperbolic geodesic ray emanating from w_0 that is tangent to $e^{i\theta}$ at w_0. $\rho_0(w_0, \Delta)$ is called the horizontal hyperbolic geodesic emanating from w_0 since its unit tangent vector at w_0 is horizontal. When w_0 and Δ are fixed, we often write ρ_θ in place of $\rho_\theta(w_0, \Delta)$. Of course, $\rho_{\theta+2n\pi} = \rho_\theta$ for all n in $\mathbb{Z}$. Let $s \mapsto w_0(s, \theta)$, $0 \le s < +\infty$, be the hyperbolic arc length parametrization of ρ_θ. This means

$$\frac{\partial w_0(s,\theta)}{\partial s} = \frac{e^{i\Theta(s,\theta)}}{\lambda_\Delta(w_0(s,\theta))}, \tag{3.1}$$

where $e^{i\Theta(s,\theta)}$ is a Euclidean unit tangent to ρ_θ at the point $w_0(s,\theta)$. Because $\Delta = \cup\{\rho_\theta : 0 \le \theta < 2\pi\}$, for each w in $\Delta \setminus \{w_0\}$ there is a unique geodesic ray ρ_θ, $0 \le \theta < 2\pi$, that contains w. Hence, there exist unique $s > 0$ and θ in $[0, 2\pi)$ with $w_0(s, \theta) = w$. The hyperbolic polar coordinates for the point w relative to the center or pole at w_0 are (s, θ). The coordinate $s = d_\Delta(w_0, w)$ is the hyperbolic distance from w_0 to w and θ is the angle between the horizontal hyperbolic geodesic ray ρ_0 and the ray ρ_θ at w_0. The hyperbolic circle with hyperbolic center w_0 and hyperbolic radius s is $c_\Delta(w_0, s) = \{w : d_\Delta(w_0, w) = s\}$. The coordinate grid for hyperbolic polar coordinates consists of hyperbolic geodesics emanating from w_0 and hyperbolic circles centered at w_0. If $f : \mathbb{D} \to \Delta$ is the Möbius transformation with $f(0) = w_0$ and $f'(0) > 0$, then $w_0(s, \theta) = f(z_0(s, \theta))$.

As we noted in the preceding section when a point in the unit disk moves away from the origin along a hyperbolic geodesic, the Euclidean distance from the origin increases and points along distinct geodesics separate monotonically in the Euclidean sense. In fact these properties hold for any disk or half-plane and for any center of hyperbolic polar coordinates.

Theorem 3.1. *Let Δ be any Euclidean disk or half-plane in $\mathbb{C}$ and $w_0 \in \Delta$.*
(a) *For each $\theta \in \mathbb{R}$, $|w_0(s,\theta) - w_0|$ is increasing for $s \geq 0$.*
(b) *For $e^{i\theta_2} \neq e^{i\theta_1}$, $|w_0(s,\theta_1) - w_0(s,\theta_2)|$ is an increasing function of $s \geq 0$.*

Proof. If $f : \mathbb{D} \to \Delta$ is a Möbius mapping with $f(0) = w_0$ and $f'(0) > 0$, then $w_0(s,\theta) = f(z_0(s,\theta))$. Suppose

$$f(z) = \frac{az+b}{cz+d},$$

where $ad - bc = 1$. Because Δ is a Euclidean disk or half-plane, ∞ does not lie in Δ. Consequently, $-d/c$, the preimage of ∞, cannot lie in $\mathbb{D}$; equivalently, $|c| \leq |d|$. Since $ad - bc = 1$, this implies $d \neq 0$. Also, $w_0 = f(0) = b/d$.

(a) If $D(s) = \log|w_0(s,\theta) - w_0| = \log|f(z_0(s,\theta)) - w_0|$, then by using (2.2) we obtain

$$\begin{aligned} D'(s) &= \operatorname{Re} \frac{f'(z_0(s,\theta))}{f(z_0(s,\theta)) - w_0} \frac{\partial z_0(s,\theta)}{\partial s} \\ &= \frac{1 - |z_0(s,\theta)|^2}{2|z_0(s,\theta)|} \operatorname{Re} \frac{z_0(s,\theta) f'(z_0(s,\theta))}{f(z_0(s,\theta)) - w_0}. \end{aligned} \tag{3.2}$$

From

$$f'(z) = \frac{1}{(cz+d)^2} \quad \text{and} \quad f(z) - w_0 = \frac{z}{d(cz+d)},$$

we obtain

$$\frac{zf'(z)}{f(z) - w_0} = \frac{d}{cz+d}.$$

Then for $z \in \mathbb{D}$

$$\operatorname{Re} \frac{zf'(z)}{f(z) - w_0} = \operatorname{Re} \frac{d\bar{c}\bar{z} + |d|^2}{|cz+d|^2} > 0 \tag{3.3}$$

because $|c| \leq |d|$ and $|z| < 1$. Thus, (3.3) and (3.2) imply $D(s)$ is increasing for $s \geq 0$, so $|w_0(s,\theta) - w_0|$ is increasing for $s \geq 0$.

(b) We assume $-\pi/2 \leq \theta_1 = -\theta < 0 < \theta_2 = \theta \leq \pi/2$; the general case can be reduced to this situation by performing a rotation. If

$$E(s) = \log|w_0(s,\theta) - w_0(s,-\theta)| = \log|f(z_0(s,\theta)) - f(z_0(s,-\theta))|,$$

then

$$E'(s) = \operatorname{Re} \frac{f'(z_0(s,\theta)) \frac{\partial z_0(s,\theta)}{\partial s} - f'(z_0(s,-\theta)) \frac{\partial z_0(s,-\theta)}{\partial s}}{f(z_0(s,\theta)) - f(z_0(s,-\theta))}.$$

Because of (2.2) and $|z_0(s,\theta)| = |z_0(s,-\theta)|$, we obtain

$$E'(s) = \frac{1 - |z_0(s,\theta)|^2}{2|z_0(s,\theta)|} \operatorname{Re} \left(\frac{z_0(s,\theta) f'(z_0(s,\theta)) - z_0(s,-\theta) f'(z_0(s,-\theta))}{f(z_0(s,\theta)) - f(z_0(s,-\theta))} \right). \tag{3.4}$$

Direct calculation produces

$$\frac{zf'(z) - \zeta f'(\zeta)}{f(z) - f(\zeta)} = \frac{d^2 - c^2 \zeta z}{(cz+d)(c\zeta+d)}.$$

Set $t = c/d$. Then

$$\frac{zf'(z) - \zeta f'(\zeta)}{f(z) - f(\zeta)} = \frac{1 - t^2\zeta z}{(1+tz)(1+t\zeta)} = \frac{1}{2}\left(\frac{1-t\zeta}{1+t\zeta} + \frac{1-tz}{1+tz}\right).$$

Because $(1-w)/(1+w)$ has positive real part for $w \in \mathbb{D}$ and $|t\zeta|, |tz| < 1$, we conclude that for all $z, \zeta \in \mathbb{D}$

$$\operatorname{Re}\left(\frac{zf'(z) - \zeta f'(\zeta)}{f(z) - f(\zeta)}\right) > 0. \tag{3.5}$$

Hence, (3.4) and (3.5) imply $E'(s) > 0$ for $s \geq 0$, so that $|w_0(s,\theta) - w_0(s,-\theta)|$ is an increasing function of $s \geq 0$. ■

4. Hyperbolic polar coordinates in simply connected regions

A region Ω in the complex plane $\mathbb{C}$ is *hyperbolic* if $\mathbb{C} \setminus \Omega$ contains at least two points. The *hyperbolic metric* on a hyperbolic region Ω is denoted by $\lambda_\Omega(w)|dw|$ and is normalized to have curvature

$$-\frac{\triangle \log \lambda_\Omega(w)}{\lambda_\Omega^2(w)} = -1.$$

If $f : \mathbb{D} \to \Omega$ is any holomorphic universal covering projection, then the density λ_Ω of the hyperbolic metric is determined from

$$\lambda_\Omega(f(z))|f'(z)| = \frac{2}{1-|z|^2}. \tag{4.1}$$

For a, b in Ω the *hyperbolic distance* between these points is

$$d_\Omega(a,b) = \inf \int_\delta \lambda_\Omega(w)|dw|,$$

where the infimum is taken over all piecewise smooth paths δ in Ω joining a and b. A path γ connecting a and b is a *hyperbolic geodesic arc* if

$$d_\Omega(a,b) = \int_\gamma \lambda_\Omega(w)|dw|.$$

A hyperbolic geodesic always exists, but need not be unique when Ω is multiply connected. Given $a \in \Omega$ and $r > 0$, $D_\Omega(a,r) = \{z \in \Omega : d_\Omega(a,z) < r\}$ is the hyperbolic disk with hyperbolic center a and hyperbolic radius r.

When Ω is simply connected, any conformal mapping $f : \mathbb{D} \to \Omega$ is an isometry from the hyperbolic metric on $\mathbb{D}$ to the hyperbolic metric on Ω. In this case f maps hyperbolic geodesics onto hyperbolic geodesics and hyperbolic disks onto hyperbolic disks. If Ω is multiply connected, then a covering f is only a local isometry, not an isometry.

Suppose Ω is a simply connected hyperbolic region, $w_0 \in \Omega$ and $f : \mathbb{D} \to \Omega$ is the unique conformal mapping with $f(0) = w_0$ and $f'(0) > 0$. We can relate hyperbolic polar coordinates on Ω with pole at w_0 to those on $\mathbb{D}$ with pole at the origin by using f. Tangent vectors for geodesic rays can be expressed in terms of this conformal mapping. Because f is an isometry from the hyperbolic metric on $\mathbb{D}$ to the hyperbolic metric on Ω and $f'(0) > 0$, $w_0(s,\theta) = f(z_0(s,\theta))$ is the hyperbolic arc length parametrization of $\rho_\theta(w_0,\Omega)$ and the tangent vector to $\rho_\theta(w_0,\Omega)$ is

$$\frac{\partial w_0(s,\theta)}{\partial s} = \frac{f'(z_0(s,\theta))e^{i\theta}}{\lambda_{\mathbb{D}}(z_0(s,\theta))}. \tag{4.2}$$

Thus,

$$\frac{\partial w_0(0,\theta)}{\partial s} = \frac{f'(0)e^{i\theta}}{2},$$

so that $s \mapsto w_0(s,\theta)$ is parallel to $e^{i\theta}$ at w_0. By making use of (4.1) we find

$$\frac{\partial w_0(s,\theta)}{\partial s} = \frac{f'(z_0(s,\theta))}{|f'(z_0(s,\theta))|}\frac{e^{i\theta}}{\lambda_\Omega(f(z_0(s,\theta)))} = \frac{e^{i(\varphi(s,\theta)+\theta)}}{\lambda_\Omega(f(z_0(s,\theta)))}, \tag{4.3}$$

where $e^{i\varphi(s,\theta)} = \frac{f'(z_0(s,\theta))}{|f'(z_0(s,\theta))|}$. If $\arg f'(z)$ denotes the unique branch of the argument of f' that vanishes at w_0, then $\varphi(s,\theta) = \arg f'(z_0(s,\theta))$. From (3.1) and (4.3) we obtain

$$e^{i\Theta(s,\theta)} = e^{i(\varphi(s,\theta)+\theta)}. \tag{4.4}$$

In a similar manner, hyperbolic disks in Ω are the images of hyperbolic disks in $\mathbb{D}$; explicitly, if $f : \mathbb{D} \to \Omega$ is a conformal map with $f(0) = w_0$, then $f(D_{\mathbb{D}}(0,r)) = D_\Omega(w_0,r)$.

5. Euclidean convex univalent functions

Several characterizations of Euclidean convex univalent functions are needed for our investigation of hyperbolic polar coordinates. We recall two classical characterizations of Euclidean convex and starlike univalent functions. First, a locally univalent holomorphic function f defined on $\mathbb{D}$ is a conformal map onto a Euclidean convex region if and only if [2, p 42]

$$1 + \operatorname{Re}\frac{zf''(z)}{f'(z)} \geq 0 \tag{5.1}$$

for $z \in \mathbb{D}$. Second, if $f(0) = w_0$, then a holomorphic function f defined on $\mathbb{D}$ maps $\mathbb{D}$ conformally onto a region starlike with respect to w_0 if and only if [2, p 41]

$$\operatorname{Re}\frac{zf'(z)}{f(z) - w_0} \geq 0 \tag{5.2}$$

for $z \in \mathbb{D}$.

Theorem 5.1. *Suppose f is holomorphic and locally univalent on $\mathbb{D}$. f is Euclidean convex univalent on $\mathbb{D}$ if and only if*

$$\operatorname{Re} \frac{zf'(z) - \zeta f'(\zeta)}{f(z) - f(\zeta)} > 0 \tag{5.3}$$

for all z, ζ in $\mathbb{D}$.

Proof. We present the short proof. Suppose f is Euclidean convex univalent on $\mathbb{D}$. Then ([27] and [30])

$$\operatorname{Re} \left\{ \frac{2zf'(z)}{f(z) - f(\zeta)} - \frac{z+\zeta}{z-\zeta} \right\} > 0 \tag{5.4}$$

for z, ζ in $\mathbb{D}$. If we interchange the roles of z and ζ in (5.4) and then add the two inequalities, we obtain

$$2 \operatorname{Re} \frac{zf'(z) - \zeta f'(\zeta)}{f(z) - f(\zeta)} > 0,$$

which is equivalent to (5.3).

Conversely, suppose (5.3) holds for all z, ζ in $\mathbb{D}$. Since

$$\lim_{\zeta \to z} \frac{zf'(z) - \zeta f'(\zeta)}{f(z) - f(\zeta)} = 1 + \frac{zf''(z)}{f'(z)},$$

we obtain (5.1). Hence, f is Euclidean convex univalent on $\mathbb{D}$. ■

Theorem 5.2. *If f is a normalized, $f(0) = 0$ and $f'(0) = 1$, Euclidean convex univalent function on $\mathbb{D}$ and $\theta \in (0, \pi/2]$, then*

$$\frac{2|z|\sin\theta}{1 + 2|z|\cos\theta + |z|^2} \leq |f(e^{i\theta}z) - f(e^{-i\theta}z)| \leq \frac{2|z|\sin\theta}{1 - |z|^2} \left(\frac{1+|z|}{1-|z|} \right)^{\cos\theta}. \tag{5.5}$$

The lower bound is best possible for all $\theta \in (0, \pi/2]$ and the upper bound is sharp for $\theta = \pi/2$.

Proof. We sketch the idea of the proof. Fix θ in $(0, \pi/2]$ and consider the function

$$g(z) = \frac{f(e^{i\theta}z) - f(e^{-i\theta}z)}{e^{i\theta} - e^{-i\theta}} = \frac{f(e^{i\theta}z) - f(e^{-i\theta}z)}{2i\sin\theta}.$$

From

$$\frac{zg'(z)}{g(z)} = \frac{e^{i\theta}zf'(e^{i\theta}z) - e^{-i\theta}zf'(e^{-i\theta}z)}{f(e^{i\theta}z) - f(e^{-i\theta}z)},$$

Theorem 5.1 implies that g is starlike with respect to the origin on $\mathbb{D}$ because (5.2) holds with $w_0 = 0$. If

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

then

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\sin n\theta}{\sin\theta} a_n z^n.$$

As f is convex univalent, $|a_2| \leq 1$ [2]. Hence,

$$\frac{|g''(0)|}{2} = \left|\frac{\sin 2\theta}{\sin\theta}\right| |a_2| \leq 2\cos\theta.$$

Then [3] gives the inequalities in (5.5). ∎

Corollary 5.3. *If f is a normalized, $f(0) = 0$ and $f'(0) = 1$, Euclidean convex univalent function on $\mathbb{D}$, then for $\varphi \in (0, \pi/2]$*

$$\frac{2r}{1+r^2} \leq |f(re^{i\varphi}) - f(-re^{i\varphi})| \leq \frac{2r}{1-r^2}. \tag{5.6}$$

These bounds are sharp.

Example 5.4. If $K(z) = z/(1-z)$, then

$$\begin{aligned} K(e^{i\theta}z) - K(e^{-i\theta}z) &= \frac{e^{i\theta}z}{1-e^{i\theta}z} - \frac{e^{-i\theta}z}{1-e^{-i\theta}z} \\ &= \frac{(2i\sin\theta)z}{1-(2\cos\theta)z+z^2}. \end{aligned}$$

This shows that the lower bound in (5.5) is sharp for $K(z)$ when $z = -r$, r is in $(0, 1)$, for any θ in $(0, \pi/2]$. For $\theta = \pi/2$ the upper bound in (5.5) is sharp for the function K when $z = ir$, r in $(0, 1)$. Also,

$$K(r) - K(-r) = \frac{2r}{1-r^2}$$

and

$$K(ir) - K(-ir) = \frac{2ir}{1+r^2},$$

so both bounds in (5.6) are sharp.

6. Euclidean convex regions

We establish various Euclidean properties for hyperbolic polar coordinates in Euclidean convex regions; in fact, these Euclidean properties characterize convex regions. Throughout this section we employ the notation of Section 4. In particular, f will always denote a conformal map of $\mathbb{D}$ onto Ω with $f(0) = w_0$ and $f'(0) > 0$. We show that for each fixed θ, the point $w_0(s, \theta)$ moves monotonically away from w_0 in the Euclidean sense. We give sharp upper and lower bounds on $|w_0(s, \theta) - w_0|$ in terms of s and $\lambda_\Omega(w_0)$. Also, in any convex region Ω distinct hyperbolic geodesic rays separate monotonically in the Euclidean sense; this means that for $e^{i\theta_2} \neq e^{i\theta_1}$, the distance $|w_0(s, \theta_1) - w_0(s, \theta_2)|$ is an increasing function of s. We give sharp upper and lower bounds on the difference $|w_0(s, \theta_1) - w_0(s, \theta_2)|$. These (and other) Euclidean properties of hyperbolic polar coordinates characterize convex regions.

For example, a classical result of Study [29] implies that for every $w_0 \in \Omega$ each hyperbolic circle $c_\Omega(w_0, s)$ is a Euclidean convex curve when Ω is convex. The result of Study asserts that if f is a Euclidean convex univalent function, then $f(\{z : |z| < r\})$ is Euclidean convex for $0 < r < 1$. Conversely, if every hyperbolic circle is Euclidean convex, then Ω is an increasing union of Euclidean convex regions and so is Euclidean convex.

Theorem 6.1. *Let Ω be a simply connected hyperbolic region in $\mathbb{C}$.*
(a) *If Ω is Euclidean convex and $w_0 \in \Omega$, then for each θ in $\mathbb{R}$, $|w_0(s,\theta) - w_0|$ is an increasing function of s and*

$$\frac{1-e^{-s}}{\lambda_\Omega(w_0)} \leq |w_0(s,\theta) - w_0| \leq \frac{e^s - 1}{\lambda_\Omega(w_0)}. \tag{6.1}$$

These bounds are best possible.
(b) *Suppose that for every w_0 in Ω and for each θ in $\mathbb{R}$, $|w_0(s,\theta) - w_0|$ is an increasing function of s. Then Ω is Euclidean convex.*

The proof of Theorem 6.1 is given in [16].

Example 6.2. For the upper half-plane $\mathbb{H}$, $\lambda_{\mathbb{H}}(w) = 1/\mathrm{Im}(w)$. Then for $w_0 = i$, $w_0(s,\pi/2) = i + i(e^s - 1)$, $w_0(s,-\pi/2) = i - i(1 - e^{-s})$ and $1/\lambda_{\mathbb{H}}(i) = 1$, so the upper and lower bounds are best possible.

Theorem 6.3. *Suppose Ω is a simply connected hyperbolic region in $\mathbb{C}$.*
(a) *If Ω is Euclidean convex, $w_0 \in \Omega$ and $e^{i\theta_2} \neq e^{i\theta_1}$, then $|w_0(s,\theta_1) - w_0(s,\theta_2)|$ is an increasing function of $s \geq 0$ and*

$$\frac{2\sin\theta\tanh s}{1+\cos\theta\tanh s} \leq |w_0(s,\theta_1) - w_0(s,\theta_2)|\lambda_\Omega(a) \leq 2e^{s\cos\theta}\sin\theta\sinh s, \tag{6.2}$$

where $\theta = (\theta_2 - \theta_1)/2$.
(b) *If for some w_0 in Ω and all $e^{i\theta_2} \neq e^{i\theta_1}$, $|w_0(s,\theta_1) - w_0(s,\theta_2)|$ is an increasing function of $s \geq 0$, then Ω is Euclidean convex.*

Proof. We sketch the proof of (a). First, by translating Ω if necessary, we may assume $w_0 = 0$. Next, by rotating Ω about the origin if needed, we may assume $-\pi/2 \leq \theta_1 = -\theta < 0 < \theta_2 = \theta \leq \pi/2$. Then

$$|w_0(s,\theta) - w_0(s,-\theta)| = |f(z(s,\theta)) - f(z(s,-\theta))|.$$

All of the quantities involved in the theorem are invariant under translation and rotation. If

$$E(s) = \log|w_0(s,\theta) - w_0(s,-\theta)| = \log|f(z(s,\theta)) - f(z(s,-\theta))|,$$

then

$$E'(s) = \mathrm{Re}\,\frac{f'(z(s,\theta))\frac{\partial z(s,\theta)}{\partial s} - f'(z(s,-\theta))\frac{\partial z(s,-\theta)}{\partial s}}{f(z(s,\theta)) - f(z(s,-\theta))}.$$

Because of (2.2) and $|z(s,\theta)| = |z(s,-\theta)|$, we obtain

$$E'(s) = \frac{1-|z(s,\theta)|^2}{2|z(s,\theta)|}\mathrm{Re}\left(\frac{z(s,\theta)f'(z(s,\theta)) - z(s,-\theta)f'(z(s,-\theta))}{f(z(s,\theta)) - f(z(s,-\theta))}\right). \tag{6.3}$$

Suppose Ω is Euclidean convex. Then f is a Euclidean convex univalent function and so (5.3) implies $E'(s) > 0$. Hence, $|w_0(s,\theta) - w_0(s,-\theta)|$ is an increasing function of $s \geq 0$. Next, we establish (6.2). The function $f/f'(0)$ is a normalized

Euclidean convex univalent function so (5.5) with $r = \tanh(s/2)$ gives the bounds (6.2) for $\theta_2 = \theta$ and $\theta_1 = -\theta$ since $f'(0) = 2/\lambda_\Omega(a)$. ■

Corollary 6.4. *Suppose $\Omega \neq \mathbb{C}$ is a Euclidean convex region and w_0 is a point of Ω. Then $|w_0(s,\theta) - w_0(s,\theta+\pi)|$ is increasing and*

$$\tanh(s) \leq |w_0(s,\theta) - w_0(s,\theta+\pi)| \frac{\lambda_\Omega(a)}{2} \leq \sinh(s). \tag{6.4}$$

Both bounds are sharp for a half-plane.

Proof. This is the special case of the theorem in which $\theta_2 = \theta + \pi$ and $\theta_1 = \theta$. It corresponds to two hyperbolic geodesic rays emanating from w_0 in opposite directions. ■

The lower bound in (6.2) has a simple geometric consequence. It gives

$$\lim_{s\to+\infty} |w_0(s,\theta_1) - w_0(s,\theta_2)| \geq \frac{2\tan(\theta/2)}{\lambda_\Omega(w_0)}.$$

For any Euclidean convex region this shows that the 'ends' of distinct hyperbolic geodesic rays emanating from w_0 cannot be too close. In particular, (6.4) implies that the two ends of a single hyperbolic geodesic cannot be closer than $2/\lambda_\Omega(w_0)$ for any point w_0 on the geodesic. This inequality is sharp for the upper half-plane; we only consider the special case in which $\theta_1 = 0$ and $\theta_2 = \pi$. Consider $w_0 = ib$, where $b > 0$. Then $\lambda_{\mathbb{H}}(w_0) = 1/b$. For the hyperbolic geodesic γ through ib that meets $\mathbb{R}$ in $\pm b$,

$$\lim_{s\to+\infty} |w_0(s,0) - w_0(s,\pi)| = 2b = \frac{2}{\lambda_{\mathbb{H}}(w_0)}.$$

7. Spherical geometry

We discuss the geometry of the *spherical plane* $\mathbb{C}_\infty$ with the chordal distance χ, the spherical metric $\sigma(z)\,|dz|$ and the induced spherical distance d_σ.

The extended complex plane $\mathbb{C}_\infty$ is sometimes called the Riemann sphere because stereographic projection transforms $\mathbb{C}_\infty$ into the unit sphere. Let $\mathbb{S}$ be the unit sphere $\{\mathbf{x} \in \mathbb{R}^3 : ||\mathbf{x}|| = 1\}$ in $\mathbb{R}^3$, and let $\mathbf{n} = (0,0,1)$ be the 'north pole'. The stereographic projection φ of $\mathbb{C}_\infty$ onto $\mathbb{S}$ is defined as follows. We regard the complex plane $\mathbb{C}$ as a subset of $\mathbb{R}^3$ by identifying $z = x + iy$ with the point $(x,y,0)$. For z in $\mathbb{C}$, the line through $z = (x,y,0)$ and $\mathbf{n}$ meets $\mathbb{S}$ at $\mathbf{n}$ and at a second point $\varphi(z)$. This defines φ on $\mathbb{C}$, and we set $\varphi(\infty) = \mathbf{n}$. It is easy to see that if $z = x + iy \in \mathbb{C}$, then

$$\varphi(x+iy) = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right).$$

Observe that $\varphi(0) = (0,0,-1)$, the south pole, and that $\varphi(z) = z = (x,y,0)$ if and only if $|z| = 1$.

The chordal distance χ is obtained by the following procedure. Use φ to transfer points in $\mathbb{C}_\infty$ to $\mathbb{S}$; then measure the Euclidean distance in $\mathbb{R}^3$ between the image points. Thus, χ is defined by

$$\chi(z,w) = ||\varphi(z) - \varphi(w)||;$$

explicitly,

$$\chi(z,w) = \frac{2|z-w|}{\sqrt{(1+|z|^2)(1+|w|^2)}}, \quad \chi(z,\infty) = \frac{2}{\sqrt{(1+|z|^2)}}.$$

This interpretation of χ immediately shows that it is a distance function on $\mathbb{C}_\infty$. Also, the metric space $(\mathbb{C}_\infty, \chi)$ is homeomorphic to $\mathbb{S}$ with the restriction of the Euclidean metric; so $(\mathbb{C}_\infty, \chi)$ is compact and connected.

The spherical metric on $\mathbb{C}_\infty$ is given by

$$\sigma(w)|dw| = \frac{2|dw|}{1+|w|^2};$$

it has curvature

$$-\frac{\Delta \log \sigma(w)}{\sigma^2(w)} = 1.$$

The spherical distance on $\mathbb{C}_\infty$ derived from this metric is

$$d_\sigma(z,w) = 2\tan^{-1}\left|\frac{z-w}{1+\bar{w}z}\right| \le \pi.$$

The chordal and spherical metrics are related to each other by the formula

$$\chi(z,w) = 2\sin\left(\frac{1}{2}d_\sigma(z,w)\right).$$

From $2\theta/\pi \le \sin\theta \le \theta$ when $0 \le \theta \le \pi/2$, we obtain

$$(2/\pi)d_\sigma(z,w) \le \chi(z,w) \le d_\sigma(z,w),$$

so the two distances induce the same topology on $\mathbb{C}_\infty$. Note that

$$\lim_{w\to z}\frac{\chi(z,w)}{|z-w|} = \sigma(z) = \lim_{w\to z}\frac{d_\sigma(z,w)}{|z-w|}.$$

We present a complete description of the isometries of the spherical plane. The orientation preserving conformal isometries of the spherical plane form a group. All of the following groups are identical,
(1) the group of conformal isometries of the chordal distance;
(2) the group of conformal isometries of the spherical distance;
(3) the group of conformal isometries of the spherical metric;
(4) the group of Möbius maps of the form

$$z \mapsto \frac{az-\bar{c}}{cz+\bar{a}}, \quad |a|^2+|c|^2 = 1.$$

The orientation-preserving isometries of $\mathbb{R}^3$ that fix the origin are the rotations of $\mathbb{R}^3$, and these are represented by the group $\mathrm{SO}(3)$ of 3×3 orthogonal matrices with determinant one. The group $\mathrm{SO}(3)$ is conjugate to the group $\varphi^{-1}\mathrm{SO}(3)\varphi$

which acts on $\mathbb{C}_\infty$; the four identical groups above are equal to $\varphi^{-1}\mathrm{SO}(3)\varphi$. For this reason the isometries of the spherical plane are sometimes called rotations.

For antipodal $z, w \in \mathbb{C}_\infty$, that is, $w = -1/\bar{z}$, any of the infinitely many great circular arcs connecting z and w is a spherical geodesic. If $z, w \in \mathbb{C}_\infty$ are not antipodal, then the unique spherical geodesic arc is the shorter arc between z and w of the unique great circle through z and w.

Just as one studies convex regions in the Euclidean plane it is natural to study convex regions in the spherical plane. A simply connected region Ω on $\mathbb{C}_\infty$ is called *spherically convex* (relative to spherical geometry on $\mathbb{C}_\infty$) if for each pair of $z, w \in \Omega$ every spherical geodesic connecting z and w also lies in Ω. If Ω is spherically convex and contains a pair of antipodal points, then $\Omega = \mathbb{C}_\infty$. A meromorphic and univalent function f defined on $\mathbb{D}$ is called *spherically convex* if its image $f(\mathbb{D})$ is a spherically convex subset of $\mathbb{C}_\infty$. A number of authors have studied spherically convex functions; for example, [6], [8], [11], [15], [19], [21] and [25].

8. Spherically convex univalent functions

In our discussion of Euclidean properties of hyperbolic geodesics, characterizations of Euclidean convex functions played a crucial role. Therefore, it is not surprising that characterizations of spherically convex functions play an important role in investigating spherical properties of hyperbolic geodesics. One such characterization obtained by Mejia and Minda [19] is

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \frac{2zf'(z)\overline{f(z)}}{1+|f(z)|^2}\right\} \geq 0 \tag{8.1}$$

for all z in $\mathbb{D}$; also see [8]. Sometimes it is difficult to use (8.1) because it contains the nonholomorphic term $2zf'(z)\overline{f(z)}/(1+|f(z)|^2)$. One way to overcome this difficulty is to establish two-variable characterizations for spherically convex functions which are holomorphic in one of the two variables and are analogous to Theorem 5.1

We now state two-variable characterizations for spherically convex functions that will be applied to investigate properties of hyperbolic polar coordinates on spherically convex regions and to derive other results for spherically convex functions.

Theorem 8.1. *Let f be meromorphic and locally univalent in $\mathbb{D}$. Then f is spherically convex if and only if*

$$\mathrm{Re}\left\{\frac{2zf'(z)}{f(z)-f(\zeta)} - \frac{z+\zeta}{z-\zeta} - \frac{2zf'(z)\overline{f(\zeta)}}{1+\overline{f(\zeta)}f(z)}\right\} > 0 \tag{8.2}$$

for all z, ζ in $\mathbb{D}$.

Proof. Here we prove only the sufficiency. Observe that (8.2) is the spherical analog of (5.4). Let

$$p(z,\zeta)=\frac{2zf'(z)}{f(z)-f(\zeta)}-\frac{z+\zeta}{z-\zeta}-\frac{2zf'(z)\overline{f(\zeta)}}{1+\overline{f(\zeta)}f(z)}. \tag{8.3}$$

We show that if f satisfies the inequality (8.2), then (8.1) holds for all $z\in\mathbb{D}$, which characterizes spherically convex functions [19]. The assumption is that $\mathrm{Re}\,\{p(z,\zeta)\}>0$ for $z,\zeta\in\mathbb{D}$. Since

$$p(z,z)=1+\frac{zf''(z)}{f'(z)}-\frac{2zf'(z)\overline{f(z)}}{1+|f(z)|^2}, \tag{8.4}$$

f is spherically convex. ∎

Corollary 8.2. *Suppose f is meromorphic and locally univalent in $\mathbb{D}$. Then f is spherically convex if and only if*

$$\mathrm{Re}\left\{\frac{zf'(z)-\zeta f'(\zeta)}{f(z)-f(\zeta)}-\frac{zf'(z)\overline{f(\zeta)}+\overline{\zeta f'(\zeta)}f(z)}{1+\overline{f(\zeta)}f(z)}\right\}>0 \tag{8.5}$$

for all z, ζ in $\mathbb{D}$.

Proof. Note that (8.5) follows from (8.2) in the same manner that (5.3) was derived from (5.4). Conversely, suppose (8.5) holds for all z, ζ in $\mathbb{D}$. Set

$$q(z,\zeta)=\frac{zf'(z)-\zeta f'(\zeta)}{f(z)-f(\zeta)}-\frac{zf'(z)\overline{f(\zeta)}+\overline{\zeta f'(\zeta)}f(z)}{1+\overline{f(\zeta)}f(z)}. \tag{8.6}$$

Then $\mathrm{Re}\,\{q(z,z)\}>0$ for all z in $\mathbb{D}$. As

$$q(z,z)=1+\frac{zf''(z)}{f'(z)}-\frac{2zf'(z)\overline{f(z)}}{1+|f(z)|^2},$$

the inequality (8.1) holds. Therefore, f is spherically convex. ∎

As we pointed out earlier, we cannot easily derive properties of spherically convex functions from (8.1) since it contains a nonholomorphic term. With Theorem 8.1, this difficulty is overcome in some cases. If f is spherically convex, then $p(z,\zeta)$ is holomorphic for in $z\in\mathbb{D}$, has positive real part, and so satisfies

$$\left|p(z,\zeta)-\frac{1+|z|^2}{1-|z|^2}\right|\le\frac{2|z|}{1-|z|^2}. \tag{8.7}$$

The nonholomorphic function $p(z,z)$ (see (8.4)) still satisfies the inequality (8.7), which holds for the well known class consisting of holomorphic functions $p(z)$ in $\mathbb{D}$ with $p(0)=1$ and $\mathrm{Re}\,\{p(z)\}>0$. Note that

$$\left|p(z,z)-\frac{1+|z|^2}{1-|z|^2}\right|\le\frac{2|z|}{1-|z|^2}$$

also characterizes spherically convex functions and implies the inequality (8.1), see [11].

This idea can be used to derive a number of results for spherically convex functions. First, recall that a holomorphic and univalent function f in $\mathbb{D}$ with $f(0) = f'(0) - 1 = 0$ is called starlike of order $\beta \geq 0$ if $\operatorname{Re}\{zf'(z)/f(z)\} > \beta$ in $\mathbb{D}$. Using Theorem 8.1, we show that spherically convex functions are closely related to starlike functions.

Theorem 8.3. *If $f(z)$ is spherically convex with $f(0) = 0$, then for every $\zeta \in \mathbb{D}$,*

$$F_\zeta(z) = \frac{z\zeta}{f(\zeta)} \frac{f(z) - f(\zeta)}{(z-\zeta)\left(1 + \overline{f(\zeta)}f(z)\right)}$$

is starlike of order $1/2$.

Proof. Direct calculations yield

$$\frac{2zF_\zeta(z)}{F_\zeta(z)} - 1 = p(z, \zeta).$$

Theorem 8.1 implies the result. ∎

Since $\operatorname{Re}\{F(z)/z\} > 1/2$ and $F(z)^2/z$ is starlike if F is starlike of order $1/2$ (see [26, p. 49]), we get the following results as corollaries of Theorem 8.3.

Corollary 8.4. *If $f(z)$ is spherically convex with $f(0) = 0$, then for every $\zeta \in \mathbb{D}$,*

$$\operatorname{Re}\left\{\frac{\zeta}{f(\zeta)} \frac{f(z) - f(\zeta)}{(z-\zeta)\left(1 + \overline{f(\zeta)}f(z)\right)}\right\} > \frac{1}{2}$$

for all z in $\mathbb{D}$.

Corollary 8.5. *If $f(z)$ is spherically convex with $f(0) = 0$, then for every $\zeta \in \mathbb{D}$,*

$$\frac{F_\zeta^2(z)}{z} = \frac{z\zeta^2}{f(\zeta)^2} \frac{(f(z) - f(\zeta))^2}{(z-\zeta)^2\left(1 + \overline{f(\zeta)}f(z)\right)^2}$$

is starlike in $\mathbb{D}$.

Mejia and Pommerenke [21] obtained a number of results for spherically convex functions by observing that $f(z)$ is (Euclidean) convex when $f(z)$ is spherically convex and $f(0) = 0$. We now provide the sharp order of Euclidean convexity for spherically convex functions that fix the origin.

Corollary 8.6. *Let $f(z) = \alpha z + a_2 z^2 + \ldots$, $0 < \alpha < 1$, be spherically convex. Then for all z in $\mathbb{D}$*

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left(\frac{\alpha}{1 + \sqrt{1 - \alpha^2}}\right)^2.$$

This result is best possible for each α.

Example 8.7. For $0 < \alpha \leq 1$, the spherical half-plane, or hemisphere, $\Omega_\alpha = \left\{w : |w - \sqrt{1-\alpha^2}/\alpha| < 1/\alpha\right\}$ is spherically convex and

$$k_\alpha(z) = \frac{\alpha z}{1 - \sqrt{1-\alpha^2} z}$$

maps $\mathbb{D}$ conformally onto Ω_α. For the function k_α,

$$\inf \left\{ \mathrm{Re} \left\{ 1 + \frac{z k_\alpha''(z)}{k_\alpha'(z)} \right\} : z \in \mathbb{D} \right\} = \left(\frac{\alpha}{1 + \sqrt{1-\alpha^2}} \right)^2 .$$

Next, we give the sharp lower bound on Re $\{a_2 f(z)\}$ for normalized spherically convex functions $f(z) = \alpha z + a_2 z^2 + \ldots$. Similar results hold for Euclidean convex functions [4] and hyperbolically convex functions [14].

Theorem 8.8. *Let $f(z) = \alpha z + a_2 z^2 + \ldots$, $0 < \alpha \leq 1$, be spherically convex. Then for all z in $\mathbb{D}$*

$$\mathrm{Re}\ \{a_2 f(z)\} \geq 1 - \alpha^2 - \sqrt{1-\alpha^2}.$$

This result is best possible for all α.

It is easy to see that for the spherically convex functions $k_\alpha(z)$, the infimum of Re $\{a_2 k_\alpha(z)\}$ over $z \in \mathbb{D}$ is $1 - \alpha^2 - \sqrt{1-\alpha^2}$, so the lower bound is sharp for all $\alpha \in (0, 1]$.

9. Spherically convex regions

Now, we establish certain properties of hyperbolic polar coordinates in spherically convex regions. It is convenient to use the density of the hyperbolic metric relative to the spherical metric; that is,

$$\mu_\Omega(w) = \frac{\lambda_\Omega(w)|dw|}{\sigma(w)|dw|} = \frac{1}{2}(1 + |w|^2)\lambda_\Omega(w).$$

Then $\lambda_\Omega(w)|dw| = \mu_\Omega(z)\sigma(w)|dw|$ and μ_Ω is invariant under all rotations of the sphere.

Theorem 9.1. *Suppose Ω is a hyperbolic region in $\mathbb{C}_\infty$.*
(a) *If Ω is spherically convex and $w_0 \in \Omega$, then $d_\sigma(w_0(s,\theta), w_0)$ is an increasing function of s for all θ in $\mathbb{R}$. Moreover, the sharp bounds*

$$\frac{\tanh(s/2)}{\mu_\Omega(w_0) + \tanh(s/2)\sqrt{\mu_\Omega^2(w_0) - 1}} \leq \tan \frac{1}{2} d_\sigma(w(s,\theta), w_0) \leq \frac{\tanh(s/2)}{\mu_\Omega(w_0) - \tanh(s/2)\sqrt{\mu_\Omega^2(w_0) - 1}}.$$

hold.
(b) *If $d_\sigma(w_0(s,\theta), w_0)$ is an increasing function of s for each w_0 in Ω and all θ in $\mathbb{R}$, then Ω is spherically convex.*

The proof of Theorem 9.1 is given in [17].

Example 9.2. Consider the function k_α defined in Example 8.7. For $w_0 = 0$, $\mu_{\Omega_\alpha}(w_0) = 1/\alpha$,

$$w_0(s,0) = \frac{\alpha \tanh(s/2)}{1 - \tanh(s/2)\sqrt{1-\alpha^2}}$$

is the hyperbolic arc length parametrization of $[0, \frac{1+\sqrt{1-\alpha^2}}{\alpha})$, and the upper bound is equal to

$$\frac{\alpha \tanh(s/2)}{1 - \tanh(s/2)\sqrt{1-\alpha^2}} = \tan \frac{1}{2} d_\sigma(w_0(s,0), 0).$$

This shows that the upper bound is sharp. Similarly,

$$w_0(s,\pi) = \frac{-\alpha \tanh(s/2)}{1 + \tanh(s/2)\sqrt{1-\alpha^2}}$$

is the hyperbolic arc length parametrization of $(\frac{-1+\sqrt{1-\alpha^2}}{\alpha}, 0]$, and the lower bound is equal to

$$\frac{\alpha \tanh(s/2)}{1 + \tanh(s/2)\sqrt{1-\alpha^2}} = \tan \frac{1}{2} d_\sigma(w_0(s,\pi), 0).$$

Hence, the lower bound is also sharp.

Theorem 9.3. *Suppose Ω is a hyperbolic region in $\mathbb{C}_\infty$.*
(a) *If Ω is spherically convex and $w_0 \in \Omega$, then $d_\sigma(w_0(s,\theta_1), w_0(s,\theta_2))$ is an increasing function of s whenever $e^{i\theta_2} \neq e^{i\theta_1}$.*
(b) *If there exists $w_0 \in \Omega$ such that $d_\sigma(w_0(s,\theta_1), w_0(s,\theta_2))$ is an increasing function of s whenever $e^{i\theta_2} \neq e^{i\theta_1}$, then Ω is spherically convex.*

The reader can consult [17] for a proof of Thorem 9.3.

Geometrically, Theorem 9.3(a) indicates that in a spherically convex region Ω, two hyperbolic geodesics starting off in different directions from a point w_0 in Ω will spread farther apart relative to the spherical distance.

If Ω is spherically convex, then so is every hyperbolic disk and conversely. This follows from the analog of Study's theorem for spherically convex functions; see [19].

10. Hyperbolic geometry

In this section, we indicate similar monotonicity properties for hyperbolic polar coordinates in hyperbolically convex regions. Because of the numerous similarities with the Euclidean and spherical cases, we present even fewer details in this situation. It is convenient to introduce the notation

$$\nu_\Omega(w) = \frac{\lambda_\Omega(w)|dw|}{\lambda_{\mathbb{D}}(w)|dw|} = \frac{1}{2}(1 - |w|^2)\lambda_\Omega(w)$$

for the density of the hyperbolic metric of a region $\Omega \subset \mathbb{D}$ relative to the background hyperbolic metric $\lambda_{\mathbb{D}}(w)|dw|$.

A simply connected region Ω in $\mathbb{D}$ is called *hyperbolically convex* (relative to the background hyperbolic geometry on $\mathbb{D}$) if for all points $z, w \in \Omega$ the arc of the hyperbolic geodesic in $\mathbb{D}$ connecting z and w also lies in Ω. A holomorphic and univalent function f defined on $\mathbb{D}$ with $f(\mathbb{D}) \subset \mathbb{D}$ is called *hyperbolically convex* if its image $f(\mathbb{D})$ is a hyperbolically convex subset of $\mathbb{D}$. Hyperbolically convex functions have been studied by a number of authors [7], [8], [12], [13], [14], [17], [20], [22]. The related concept of hyperbolically 1-convex functions was investigated in [9].

There are known characterizations of hyperbolically convex functions. For example, a holomorphic and locally univalent function f with $f(\mathbb{D}) \subset \mathbb{D}$ is hyperbolically convex if and only if [12]

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}+\frac{2zf'(z)\overline{f(z)}}{1-|f(z)|^2}\right\} \geq 0 \tag{10.1}$$

for all z in $\mathbb{D}$. Mejia and Pommerenke [22] (also see [14]) showed that a holomorphic and locally univalent function f with $f(\mathbb{D}) \subset \mathbb{D}$ is hyperbolically convex if and only if

$$\operatorname{Re}\left\{\frac{2zf'(z)}{f(z)-f(\zeta)}-\frac{z+\zeta}{z-\zeta}+\frac{2zf'(z)\overline{f(\zeta)}}{1-\overline{f(\zeta)}f(z)}\right\} > 0 \tag{10.2}$$

for all z, ζ in $\mathbb{D}$. This is the hyperbolic analog of (8.2). Similar to the proof of Corollary 8.2, we obtain the following characterization from (10.2).

Theorem 10.1. *A holomorphic and locally univalent function f with $f(\mathbb{D}) \subset \mathbb{D}$ is hyperbolically convex if and only if*

$$\operatorname{Re}\left\{\frac{zf'(z)-\zeta f'(\zeta)}{f(z)-f(\zeta)}+\frac{zf'(z)\overline{f(\zeta)}+\overline{\zeta f'(\zeta)}f(z)}{1-\overline{f(\zeta)}f(z)}\right\} > 0 \tag{10.3}$$

for all z, ζ in $\mathbb{D}$.

These two-point characterizations can be used to derive monotonicity properties of hyperbolic polar coordinates on hyperbolically convex regions in $\mathbb{D}$.

Theorem 10.2. *Let $\Omega \subset \mathbb{D}$.*
(a) *If Ω is hyperbolically convex and $w_0 \in \Omega$, then $d_{\mathbb{D}}(w_0(s,\theta), w_0)$ is an increasing function of s for all θ in $\mathbb{R}$. Moreover, we have the following sharp bounds:*

$$\begin{aligned}
&\frac{2\tanh(s/2)}{\nu_\Omega(w_0)(1+\tanh(s/2))+\sqrt{\nu_\Omega^2(w_0)(1+\tanh(s/2))^2-4\tanh(s/2)}}\\
&\qquad\leq \tanh\frac{1}{2}d_{\mathbb{D}}(w_0(s,\theta),w_0)\\
&\leq \frac{2\tanh(s/2)}{\nu_\Omega(w_0)(1-\tanh(s/2))+\sqrt{\nu_\Omega^2(w_0)(1-\tanh(s/2))^2+4\tanh(s/2)}}.
\end{aligned}$$

(b) *If $d_{\mathbb{D}}(w_0(s,\theta), w_0)$ is an increasing function of s for each w_0 in Ω and all θ in $\mathbb{R}$, then Ω is hyperbolically convex.*

Example 10.3. For $0 < \alpha \leq 1$, the hyperbolic half-plane

$$H_\alpha = \mathbb{D} \setminus \left\{ w : \left| w + \frac{1}{\alpha} \right| \leq \frac{\sqrt{1-\alpha^2}}{\alpha} \right\}$$

is hyperbolically convex and

$$K_\alpha(z) = \frac{2\alpha z}{1 - z + \sqrt{(1-z)^2 + 4\alpha^2 z}}$$

maps $\mathbb{D}$ conformally onto H_α. When $w_0 = 0$, $\nu_{H_\alpha}(0) = 1/\alpha$, $w_0(s,0) = K_\alpha(\tanh(s/2))$ is the hyperbolic arc length parametrization of $[0,1)$, and the upper bound is equal to

$$\frac{2\alpha \tanh(s/2)}{1 - \tanh(s/2) + \sqrt{(1-\tanh(s/2))^2 + 4\alpha^2 \tanh(s/2)}} = \tanh \frac{1}{2} d_{\mathbb{D}}(w_0(s,0), 0).$$

This shows that the upper bound is sharp. Similarly, $w_0(s,\pi) = k_\alpha(-\tanh(s/2))$ is the hyperbolic arc length parametrization of $(\frac{-1-\sqrt{1-\alpha^2}}{\alpha}, 0]$, and the lower bound is equal to

$$\frac{2\alpha \tanh(s/2)}{1 + \tanh(s/2) + \sqrt{(1+\tanh(s/2))^2 - 4\alpha^2 \tanh(s/2)}} = \tanh \frac{1}{2} d_{\mathbb{D}}(w_0(s,\pi), 0).$$

Thus, the lower bound is also sharp.

The proof of Theorem 10.4 below is analogous to the proof of Theorem 9.3; the characterization (10.3) for hyperbolically convex functions is used in place of (8.2).

Theorem 10.4. *Suppose $\Omega \subset \mathbb{D}$.*
(a) *If Ω is hyperbolically convex and $w_0 \in \Omega$, then $d_{\mathbb{D}}(w_0(s,\theta_1), w_0(s,\theta_2))$ is an increasing function of s for all $e^{i\theta_2} \neq e^{i\theta_1}$.*
(b) *If there exists $w_0 \in \Omega$ such that $d_{\mathbb{D}}(w_0(s,\theta_1), w_0(s,\theta_2))$ is an increasing function of s whenever $e^{i\theta_2} \neq e^{i\theta_1}$, then Ω is hyperbolically convex.*

$\Omega \subset \mathbb{D}$ is hyperbolically convex if and only if every hyperbolic disk $D_\Omega(w_0, r)$ is hyperbolically convex as a subset of $\mathbb{D}$. This is a direct consequence of the analog of Study's Theorem for hyperbolically convex functions; see [20] and [12].

11. Concluding remarks

Relative to the background geometry (hyperbolic, Euclidean, or spherical) hyperbolic geodesics and hyperbolic disks have similar behavior in convex regions. Moreover, there are numerous similarities between conformal maps of the unit disk onto convex regions in each of the three geometries, although formulas for spherical or hyperbolic convexity can be more complicated than those for Euclidean convexity because the spherical plane and the hyperbolic plane have nonzero curvature. It is possible to obtain results for Euclidean convexity as a limit of corresponding results for spherical convexity; for example, see

Kim-Minda [6] for an illustration of the method. In the same manner Euclidean results can be obtained as the limit of hyperbolic convexity results.

Holomorphic functions defined on the unit disk can be viewed as maps from $\mathbb{D}$ to the Euclidean plane. Bounded holomorphic functions on the unit disk can be regarded as maps from $\mathbb{D}$ to the hyperbolic plane, provided they are scaled to be bounded by one. Finally, meromorphic functions on $\mathbb{D}$ can be considered as maps into the spherical plane. Sometimes connections between classical results can be made by adopting this geometric view. This paper showed the close connection between Euclidean convexity, spherical convexity and hyperbolic convexity. Also, by adopting this geometric viewpoint it is possible to recognize there should be analogs of classical results for holomorphic functions for maps into the other two geometries.

Some other function theory papers that relate to comparisons between hyperbolic, Euclidean and spherical geometry are [5], [23], [24], [25].

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