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Metrics and quasiregular mappings

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Abstract. This series of lectures intends to provide a gateway to some selected topics of quasiconformal and quasiregular maps, in particular to the main themes of [Vu1] and [Vu3]. Some of the basic notions and tools are briefly reviewed. Several problems, exercises and open problems are given throughout the text. At the end of the paper a short list of some generic open problems is presented for metric spaces, which allow a great number of variations in specific cases.

Keywords. quasiregular mappings, metric spaces.

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Contents

1. Introduction

The goal of these lectures is to provide an introduction to some of the main properties of quasiconformal and quasiregular mappings. One of the central themes here will be to study how these mappings deform distances and metrics and therefore it is natural to study our mappings between metric spaces. In most cases, the metrics will have some useful invariance or quasi-invariance properties under a set Γ of transformations, called rigid motions. An important example is the unit ball of $\mathbb{R}^n$ equipped with the hyperbolic metric in which case we may take the set Γ to be the group of the Möbius self-mappings of the ball.

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The material is largely drawn from [Vu3] and [AVV2]. In order to give the reader a chance to enter gradually this territory of mathematical research, problems of varying level are given, from easy exercises to research problems. Many more can be found in [Vu3] and [AVV2] (the exercises in [AVV2] come with solutions). Some research problems are collected at the end of the paper. Because of limitations of space, most of the details/proofs are omitted with the general reference to [V1] and [Vu3].

The idea of using invariance with respect to rigid motion to study function theory is very old. In fact, it can be traced back to nineteenth century, in particular, to the work of F. Klein. Perhaps the most natural notion of invariance is conformal invariance under the group of conformal self-maps of a given simply-connected domain. Several conformal invariants emerged from the studies of H. Grötzsch, L. Ahlfors, and A. Beurling.

A pair (X, d) is called a metric space if $X \neq \emptyset$ and $d\colon X \times X \to [0, \infty)$ satisfies the following four conditions

$$(1.1) \qquad \begin{cases} \text{(M1)} & d(x,y) \geq 0 \text{ for all } x, y \in X\,, \\ \text{(M2)} & d(x,y) = 0 \text{ iff } x = y\,, \\ \text{(M3)} & d(x,y) = d(y,x) \text{ for all } x, y \in X\,, \\ \text{(M4)} & d(x,y) \leq d(x,z) + d(z,y) \text{ for all } x, y, z \in X\,. \end{cases}$$

Let (X, d_1) and (Y, d_2) be metric spaces and let $f : X \to Y$ be a continuous mapping. Then we say that f is uniformly continuous if there exists an increasing continuous function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ and $d_2(f(x), f(y)) \leq \omega(d_1(x, y))$ for all $x, y \in X$. We call the function ω the modulus of continuity of f. If there exist $C, \alpha > 0$ such that $\omega(t) \leq Ct^\alpha$ for all $t > 0$, we say that f is Hölder-continuous with Hölder exponent α. If $\alpha = 1$, we say that f is Lipschitz with the Lipschitz constant C or simply C-Lipschitz. If f is a homeomorphism and both f and f^{-1} are C-Lipschitz, then f is C-bilipschitz or C-quasiisometry and if $C = 1$ we say that f is an isometry. These conditions are said to hold locally, if they hold for each compact subset of X.

1.2. Exercise. If $h : [0, \infty) \to [0, \infty)$ is a function and $h(t)/t$ is decreasing, show that $h(x + y) \leq h(x) + h(y)$ for all $x, y > 0$. In particular, show that if (X, d) is a metric space, then also $(X, d^\alpha), \alpha \in (0, 1)$, is.

1.3. Exercise. Let $f : [0, \infty) \to [0, \infty)$ be Hölder-continuous with exponent $\beta > 1$. Show that $f(x) = f(0)$ for all $x > 0$.

1.4. Example. Let $f : \mathbb{R}^n \to \mathbb{R}^n, f(x) = |x|^{\alpha-1}x, f(0) = 0$. Then f is Hölder-continuous with exponent α.

In most examples below, the metric spaces will have some additional structure. The metrics will often have some quasiinvariance properties. For instance, we say that a pair of metric spaces $(X_j, d_j), j = 1, 2,$ is quasiinvariant under a set Γ of mappings $f : (X_1, d_1) \to (X_2, d_2)$ if there exists $C \geq 1$ such that $1/C \leq d_2(f(x), f(y))/d_1(x, y) \leq C$ for all $x, y \in X_1, x \neq y$ and all $f \in \Gamma$. In particular, we will study metric spaces (X, d) where the group Γ of automorphisms of X acts transitively (i.e. given $x, y \in X$ there exists $h \in \Gamma$ such that $hx = y$.) If $C = 1$, then we say that d is invariant.

1.5. Examples. (1) The euclidean space $\mathbb{R}^n$ equipped with the usual metric $|x - y| = (\sum_{j=1}^n (x_j - y_j)^2)^{1/2}$, Γ is the group of translation in $\mathbb{R}^n$.

(2) The unit sphere $S^n = \{z \in \mathbb{R}^{n+1} : |z| = 1\}$ equipped with the metric of $\mathbb{R}^{n+1}$ and Γ is the set of rotations of S^n.

(3) Let $G \subset \mathbb{R}^n, G \neq \mathbb{R}^n$, for $x, y \in G$ set

$$j_G(x, y) = \log\left(1 + \frac{|x - y|}{\min\{d(x, \partial G), d(y, \partial G)\}}\right).$$

Then one can prove that j_G is a metric (this is a folklore result, see e.g. [S]). In fact, there exists a constant $C > 1$ such that for the unit ball $\mathbf{B}^n$ of $\mathbb{R}^n$

$$1/C \le \rho_{\mathbf{B}^n}(x, y)/j_{\mathbf{B}^n}(x, y) \le C$$

for all $x, y \in \mathbf{B}^n, x \neq y$. Here $\rho_{\mathbf{B}^n}$ is the hyperbolic metric of $\mathbf{B}^n$ and it is invariant under the group of Möbius self-mappings of $\mathbf{B}^n$. For the definition of $\rho_{\mathbf{B}^n}$ see below or [Vu3, Section2].

A basic geometric object of a metric space (X, d) is the ball $B_X(z, r) = \{x \in X : d(x, z) < r\}$. In order to study how balls and their boundary spheres are deformed under homeomorphisms, we introduce a deformation measure $H_f(x_0, r)$ of a ball under a homeomorphism $f : (X_1, d_1) \to (X_2, d_2)$ at a point $x \in X_1$

$$H_f(x, r) = \sup\left\{\frac{d_2(f(x), f(y))}{d_2(f(x), f(z))} : d_1(z, x) = d_1(y, x) = r\right\}.$$

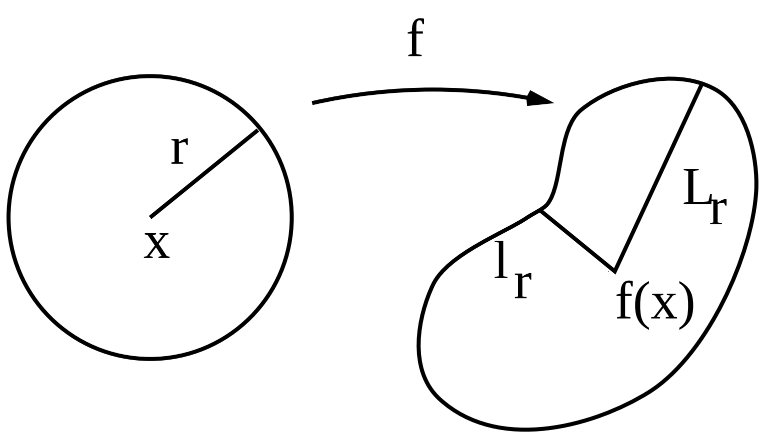

FIGURE 1. $H_f(x, r)$.

If f maps spheres centered at x onto spheres centered at $f(x)$, then $H_f(x, r) = 1$. For instance the above radial mapping $x \mapsto |x|^{\alpha-1}x$ has the property $H_f(0, r) = 1$ for all $r > 0$. Recall from Complex Analysis that a conformal map $f : D_1 \to D_2, D_j \subset \mathbb{C}, j = 1, 2$, satisfies $\lim_{r \to 0} H_f(x, r) = 1$ for all $x \in D_1$. We say that a homeomorphism $f : (X_1, d_1) \to (X_2, d_2)$ is quasiconformal (with respect to (d_1, d_2)), if there exists $C \in [1, \infty)$ such that for all $x \in X_1$

$$H_f(x) = \limsup_{r \to 0} H_f(x, r) \le C \,.$$

If f is L-bilipschitz, then f satisfies the above condition with $C = L^2$.

Let $G_j \subset \mathbb{R}^n, j = 1, 2,$ be domains and let $f : G_1 \to G_2$ be a homeomorphism. Suppose now that there exists a constant $C \ge 1$ such that for all subdomains $D \subset G_1$ the mapping $f|D : (D, j_D) \to (f(D), j_{f(D)})$ is C-Lipschitz. Fix $x_0 \in G_1$ and $r \in (0, d(x_0, \partial G_1)/2)$. If $|x - x_0| = |y - x_0| = r$ and $G = B^n(x_0, 2r) \setminus \{x_0\}$, then $j_G(x, y) \le \log 3$ and we obtain by the above C-Lipschitz-property

$$\left| \log \frac{|f(x) - f(x_0)|}{|f(y) - f(x_0)|} \right| \le j_{fG}(f(x), f(y)) \le C j_G(x, y) \le C \log 3 \,,$$

and hence $H_f(x_0) \le 3^C$, where we used the triangle inequality (Lemma 3.21 (3) below) and the fact that $x_0 \in \partial G$. Now d_1 and d_2 are the usual metrics. Thus we see that our map is quasiconformal.

In this argument the fact that $x_0 \in \partial G$ played a key role. For most of the metrics that we will consider here, even one single boundary point will be important. Most of the metrics will also be monotone with respect to the domain. Thus, for instance, if $G_1 \subset G_2 \subset \mathbb{R}^n$ are domains, then $j_{G_1}(x, y) \ge j_{G_2}(x, y)$ for all $x, y \in G_1$ and for a fixed $x_0 \in G_1$, $j_{G_1}(x_0, x) \to \infty$ as $x \to x_1 \in \partial G$. In the above argument, it was assumed that $f|D$ is Lipschitz continuous for all subdomains D of G_1 but we only used this property for subdomains of the form $B^n(x, r) \setminus \{x\}, x \in G_1$. In order to motivate this condition let us recall that a conformal map is conformal also in every subdomain.

Here we have studied the metric j, mainly because it is very easy to define and because it well represents the metrics we study here. There are now several questions:

(a) Can we characterize the class of quasiisometries or isometries in the above sense?

(b) Can we prove similar results for other metrics (and what are these metrics)?

(c) Can we say more for the case when the domains are "nice" (for instance quasidisks)?

Conformal invariants and conformally invariant metrics have been an important topic in geometric function theory during the past century. One of the first promoters of these ideas was F. Klein. In the context of quasiconformal mappings these ideas emerged as a result of the pioneering works of H. Grötzsch, O. Teichmüller, L. Ahlfors and A. Beurling on quasiconformal maps in plane domains [LV], [K]. Extension to higher dimensions is due to F.W. Gehring and J. Väisälä [G] , [V1]. The case of metric measure spaces has been studied recently by J. Heinonen, P. Koskela and many other people [H].

In the setup presented here, the aforementioned questions (a)-(c) were studied already in [Vu1] and [Vu3]. But only very few answers are known, see [H1], [H2], [HI]. These questions could also be investigated for some particular classes of domains, which would bring a very wide spectrum of new questions into play. Some examples of such classes of domains would be uniform domains and quasiconformal balls. As we will see, there still are numerous open problems in this area. It is assumed that the reader is familiar with some basic facts and definitions of the theory of quasiconformal and quasiregular maps [V1], [Vu3].

2. Möbius transformations

For $x \in \mathbb{R}^n$ and $r > 0$ let

$$\begin{aligned} B^n(x,r) &= \{ z \in \mathbb{R}^n : |x-z| < r \}, \\ S^{n-1}(x,r) &= \{ z \in \mathbb{R}^n : |x-z| = r \} \end{aligned}$$

denote the ball and sphere, respectively, centered at x with radius r. The abbreviations $B^n(r) = B^n(0,r)$, $S^{n-1}(r) = S^{n-1}(0,r)$, $\mathbf{B}^n = B^n(1)$, $S^{n-1} = S^{n-1}(1)$ will be used frequently. For $t \in \mathbb{R}$ and $a \in \mathbb{R}^n \setminus \{0\}$ we denote

$$P(a,t) = \{ x \in \mathbb{R}^n : x \cdot a = t \} \cup \{\infty\}.$$

Then $P(a,t)$ is a hyperplane in $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ perpendicular to the vector a, at distance $t/|a|$ from the origin.

2.1. Definition. Let D and D' be domains in $\mathbb{R}^n$ and let $f \colon D \to D'$ be a homeomorphism. We call f *conformal* if (1) $f \in \mathcal{C}^1$, (2) $J_f(x) \neq 0$ for all $x \in D$, and (3) $|f'(x)h| = |f'(x)||h|$ for all $x \in D$ and all $h \in \mathbb{R}^n$. If D and D' are domains in $\overline{\mathbb{R}}^n$, we call a homeomorphism $f \colon D \to D'$ *conformal* if the restriction of f to $D \setminus \{\infty, f^{-1}(\infty)\}$ is conformal.

2.2. Examples. Some basic examples of conformal mappings are the following elementary transformations.

(1) A reflection in $P(a,t)$:

$$f_1(x) = x - 2(x \cdot a - t)\frac{a}{|a|^2}\,,\ f_1(\infty) = \infty\,.$$

(2) An inversion (reflection) in $S^{n-1}(a,r)$:

$$f_2(x) = a + \frac{r^2(x-a)}{|x-a|^2}\,,\ f_2(a) = \infty\,,\ f_2(\infty) = a\,.$$

(3) A translation $f_3(x) = x + a\,,\ a \in \mathbb{R}^n\,,\ f_3(\infty) = \infty$.

(4) A stretching by a factor $k > 0$: $f_4(x) = kx\,,\ f_4(\infty) = \infty$.

(5) An orthogonal mapping, i.e. a linear map f_5 with

$$|f_5(x)| = |x|\,,\ f_5(\infty) = \infty\,.$$

2.3. Remarks. (1) The translation $x \mapsto x+a$ can be written as a composition of reflections in $P(a,0)$ and $P(a,\frac{1}{2}|a|^2)$. The stretching $x \mapsto kx$, $k>0$, can be written as a composition of inversions in $S^{n-1}(0,1)$ and $S^{n-1}(0,\sqrt{k}\,)$. It can be proved, that an orthogonal mapping can be composed of at most $n+1$ reflections in planes (see [BE, p. 23, Theorem 3.1.3]).

(2) It is easy to show that $f_1(f_1(x))=x$ and $f_2(f_2(x))=x$ for all $x\in\mathbb{R}^n$, i.e. f_1 and f_2 are involutions.

(3) It can also be shown that we have the difference formula

$$|f_2(x)-f_2(y)| = \frac{r^2|x-y|}{|x-a||y-a|}$$

for all $x,y\in\mathbb{R}^n\setminus\{a\}$.

(4) If $a_n=0$, then one can show that $f_2(\mathbf{H}^n)=\mathbf{H}^n$ and that for all $x,y\in\mathbf{H}^n$

$$\frac{|f_2(x)-f_2(y)|^2}{(f_2(x))_n(f_2(y))_n} = \frac{|x-y|^2}{(x)_n(y)_n}.$$

(5) Reflections and inversions are sense-reversing. The composition of two sense-reversing maps is sense-preserving.

2.4. Definition. A homeomorphism $f\colon\overline{\mathbb{R}}^n\to\overline{\mathbb{R}}^n$ is called a *Möbius transformation* if $f=g_1\circ\cdots\circ g_p$ where each g_j is one of the elementary transformations in 2.2(1)–(5) and p is a positive integer. Equivalently (see 2.3) f is a Möbius transformation if $f=h_1\circ\cdots\circ h_m$ where each h_j is a reflection in a sphere or in a hyperplane and m is a positive integer. If $G\subset\overline{\mathbb{R}}^n$ the set of all (sense-preserving) Möbius transformations mapping G onto itself is denoted by $\mathcal{GM}(G)$ ($\mathcal{M}(G)$).

It will be convenient to identify $\overline{\mathbb{R}}^n$ with the subset $\{\,x\in\mathbb{R}^n : x_{n+1}=0\,\}\cup\{\infty\}$ of $\overline{\mathbb{R}}^{n+1}$. The identification is given by the embedding

$$x\mapsto\tilde{\tilde{x}}=(x_1,\ldots,x_n,0)\;;\quad x=(x_1,\ldots,x_n)\in\mathbb{R}^n\,. \tag{2.5}$$

We are now going to describe a natural two–step way of extending a Möbius transformation of $\mathbb{R}^n$ to a Möbius transformation of $\mathbb{R}^{n+1}$. First, if f in $\mathcal{GM}(\overline{\mathbb{R}}^n)$ is a reflection in $P(a,t)$ or in $S^{n-1}(a,r)$, let $\tilde{\tilde{f}}$ be a reflection in $P(\tilde{\tilde{a}},t)$ or $S^n(\tilde{\tilde{a}},r)$, respectively. Then if $x\in\overline{\mathbb{R}}^n$ and $y=f(x)$, by 2.2(1)–(2) we get

$$\tilde{\tilde{f}}(x_1,\ldots,x_n,0)=(y_1,\ldots,y_n,0)=\widetilde{\widetilde{f(x)}}\,. \tag{2.6}$$

By (2.6) we may regard $\tilde{\tilde{f}}$ as an extension of f. Note that $\tilde{\tilde{f}}$ preserves the plane $x_{n+1}=0$ and each of the half–spaces $x_{n+1}>0$ and $x_{n+1}<0$. These facts follow from the formulae 2.2(1)–(2). Second, if f is an arbitrary mapping in $\mathcal{GM}(\overline{\mathbb{R}}^n)$ it has a representation $f=f_1\circ\cdots\circ f_m$ where each f_j is a reflection in a plane or a sphere. Then $\tilde{\tilde{f}}=\tilde{\tilde{f}}_1\circ\cdots\circ\tilde{\tilde{f}}_m$ is the extension of f, and it preserves the half–spaces $x_{n+1}>0$, $x_{n+1}<0$, and the plane $x_{n+1}=0$. In conclusion, every f in $\mathcal{GM}(\overline{\mathbb{R}}^n)$ has an extension $\tilde{\tilde{f}}$ in $\mathcal{GM}(\overline{\mathbb{R}}^{n+1})$. It follows from [BE, p. 31, Theorem 3.2.4] that such an extension $\tilde{\tilde{f}}$ of f is unique. The mapping $\tilde{\tilde{f}}$ is called the *Poincaré extension* of f. In the sequel we shall write x, f instead of $\tilde{\tilde{x}}$, $\tilde{\tilde{f}}$, respectively.

Many properties of plane Möbius transformations hold for n–dimensional Möbius transformations as well. The fundamental property that spheres of $\overline{\mathbb{R}}^n$ (which are spheres or planes in $\mathbb{R}^n$, see [Vu1, Exercise 1.26, p.8]) are preserved under Möbius transformations is proved in [BE, p. 28, Theorem 3.2.1].

2.7. Stereographic projection. The *stereographic projection* $\pi\colon \overline{\mathbb{R}}^n \to S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ is defined by

$$\pi(x) = e_{n+1} + \frac{x - e_{n+1}}{|x - e_{n+1}|^2}\,,\ x \in \mathbb{R}^n\ ;\ \ \pi(\infty) = e_{n+1} \tag{2.8}$$

Then π is the restriction to $\overline{\mathbb{R}}^n$ of the inversion in $S^n(e_{n+1}, 1)$. In fact, we can identify π with this inversion. Because $f^{-1} = f$ for every inversion f, it follows that π maps the "Riemann sphere" $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ onto $\overline{\mathbb{R}}^n$.

The *spherical (chordal) metric* q in $\overline{\mathbb{R}}^n$ is defined by

$$q(x,y) = |\pi(x) - \pi(y)|\ ;\ \ x,\, y \in \overline{\mathbb{R}}^n\ , \tag{2.9}$$

where π is the stereographic projection (2.8). From the definition (2.8) by calculating we obtain

$$\begin{cases} q(x,y) = \dfrac{|x-y|}{\sqrt{1+|x|^2}\,\sqrt{1+|y|^2}}; & x \neq \infty \neq y, \\[2ex] q(x,\infty) = \dfrac{1}{\sqrt{1+|x|^2}}. \end{cases} \tag{2.10}$$

For $x \in \mathbb{R}^n \setminus \{0\}$ the *antipodal* (diametrically opposite) *point* $\tilde{x}$, is defined by

$$\tilde{x} = -\frac{x}{|x|^2} \tag{2.11}$$

and we set $\tilde{\infty} = 0$, $\tilde{0} = \infty$. Then, by (2.10), $q(x, \tilde{x}\,) = 1$ and hence $\pi(x)$,$\pi(\tilde{x}\,)$ are indeed diametrically opposite points on the Riemann sphere.

2.12. Balls in the spherical metric. For $x \in \overline{\mathbb{R}}^n$ and $r \in (0,1)$ we define the spherical ball

$$Q(x,r) = \{\, z \in \overline{\mathbb{R}}^n : q(x,z) < r \,\}\ . \tag{2.13}$$

Its boundary sphere is denoted by $\partial Q(x,r)$. From the Pythagorean theorem it follows that (cf. (2.11))

$$Q(x,r) = \overline{\mathbb{R}}^n \setminus \overline{Q}\big(\tilde{x}, \sqrt{1-r^2}\,\big)\ . \tag{2.14}$$

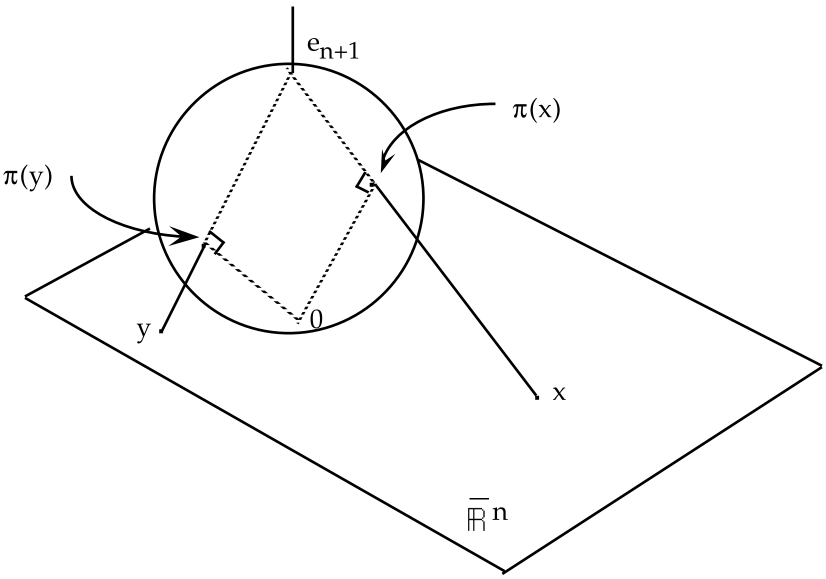

FIGURE 2. Formulae (2.8) and (2.9) visualized.

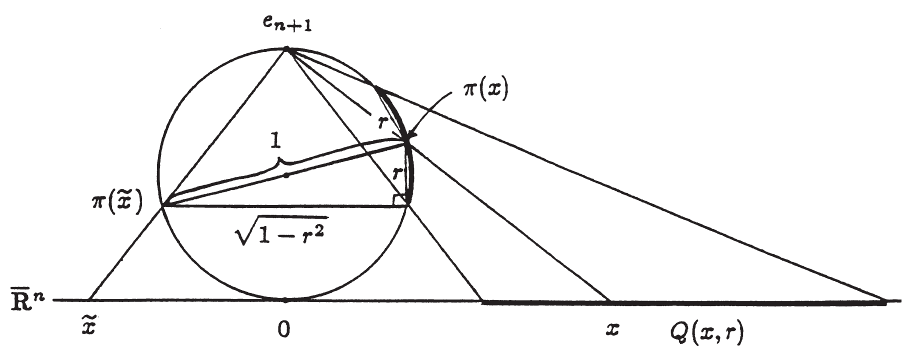

FIGURE 3. A cross-section of the Riemann sphere.

To gain insight into the geometry of spherical balls $Q(x,r)$ it is convenient to study the image $\pi Q(x,r)$ under the stereographic projection π (see figure 3). Indeed, by definition (2.9) we see that

$$\pi Q(x,r) = B^{n+1}\big(\pi(x),r\big) \cap S^n(\tfrac{1}{2}e_{n+1},\tfrac{1}{2}). \tag{2.15}$$

Either by this formula or more directly by the definition of the spherical metric (plus the fact that Möbius transformations preserve spheres) we see that in the euclidean geometry, $Q(x,r)$ is a point set of one of the following three kinds

(a) an open ball $B^n(u,s)$,
(b) the complement of $\overline{B}^n(v,t)$ in $\mathbb{R}^n$,
(c) a half–space of $\mathbb{R}^n$.

Clearly, $\partial Q(x,r)$ is either a sphere or a hyperplane of $\mathbb{R}^n$. Formula (2.14) shows, in particular, that $\pi Q(x,1/\sqrt{2}\,)$ is a half–sphere of the Riemann sphere $S^n(\frac{1}{2}e_{n+1},\frac{1}{2})$.

2.16. Absolute ratio. For an ordered quadruple a,b,c,d of distinct points in $\overline{\mathbb{R}}^n$ we define the *absolute (cross) ratio* by

$$|\,a,b,c,d\,| = \frac{q(a,c)\,q(b,d)}{q(a,b)\,q(c,d)}\,. \tag{2.17}$$

It follows from (2.10) that for distinct a,b,c,d in $\mathbb{R}^n$

$$|\,a,b,c,d\,| = \frac{|a-c|\,|b-d|}{|a-b|\,|c-d|}\,.$$

One of the most important properties of Möbius transformations is that they *preserve absolute ratios*, i.e. if $f\in\mathcal{GM}$, then

$$|\,f(a),f(b),f(c),f(d)\,| = |\,a,b,c,d\,| \tag{2.18}$$

for all distinct a,b,c,d in $\overline{\mathbb{R}}^n$. As a matter of fact, the preservation of absolute ratios is a characteristic property of Möbius transformations. It is proved in [BE, p. 72, Theorem 3.2.7] that a mapping $f\colon\overline{\mathbb{R}}^n\to\overline{\mathbb{R}}^n$ is a Möbius transformation if and only if f preserves all absolute ratios.

2.19. Automorphisms in $\mathbf{B}^n$. We shall give a canonical representation for the maps in $\mathcal{M}(\mathbf{B}^n)$. Assume that f is in $\mathcal{M}(\mathbf{B}^n)$ and that $f(a)=0$ for some $a\in\mathbf{B}^n$. We denote

$$a^* = \frac{a}{|a|^2}\,,\ \ a\in\mathbb{R}^n\setminus\{0\} \tag{2.20}$$

and $0^*=\infty$, $\infty^*=0$. Fix $a\in\mathbf{B}^n\setminus\{0\}$. Let

$$\sigma_a(x) = a^* + r^2(x-a^*)^*\,,\ r^2=|a|^{-2}-1 \tag{2.21}$$

be an inversion in the sphere $S^{n-1}(a^*,r)$ orthogonal to S^{n-1}. Then $\sigma_a(a)=0$, $\sigma_a(a^*)=\infty, \sigma_a(\mathbf{B}^n)=\mathbf{B}^n$.

Let p_a denote the reflection in the $(n-1)$–dimensional plane $P(a,0)$ through the origin and orthogonal to a and define a sense–preserving Möbius transformation by $T_a = p_a\circ\sigma_a$. Then, by (2.21), $T_a\mathbf{B}^n=\mathbf{B}^n$, $T_a(a)=0$, and with $e_a=a/|a|$ we have $T_a(e_a)=e_a$, $T_a(-e_a)=-e_a$. For $a=0$ we set $T_0=\mathrm{id}$, where id stands for the identity map. The proof of the following fundamental fact can be found in [A, p. 21], [BE, p. 40, Theorem 3.5.1].

We now define a spherical isometry t_z in $\mathcal{M}(\mathbb{R}^n)$ which maps a given point $z\in\mathbb{R}^n$ to 0 as follows. For $z=0$ let $t_z=\mathrm{id}$ and for $z=\infty$ let $t_z=p\circ f$, where f is inversion in S^{n-1} and p is reflection in the $(n-1)$–dimensional plane $x_1=0$. For $z\in\mathbb{R}^n\setminus\{0\}$ let s_z be inversion in $S^{n-1}(-z/|z|^2,r)$, where $r=\sqrt{1+|z|^{-2}}$. According to [Vu1, (1.45)], the inversion s_z is a spherical isometry and it is easy to show that $s_z(z)=0$. Let p_z be reflection in the plane $P(z,0)$. Defining

$$t_z = p_z\circ s_z\,, \tag{2.22}$$

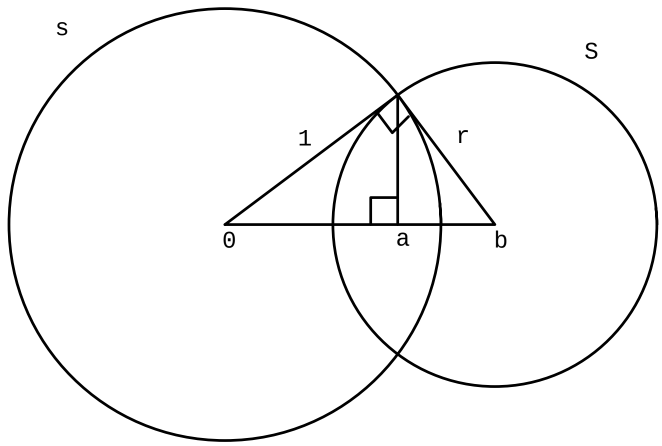

FIGURE 4. Inversion in S^{n-1}, $b = a^*$.

we see that $t_z \in \mathcal{M}(\mathbb{R}^n)$ is a spherical isometry with $t_z(z) = 0$. Hence

$$\begin{aligned} t_z(Q(z,r)) &= Q(0,r) = B^n\big(r/\sqrt{1-r^2}\,\big), \\ |t_z(x)|^2 &= \frac{q(x,z)^2}{1-q(x,z)^2} \end{aligned} \tag{2.23}$$

for all $x, z \in \mathbb{R}^n$, $r \in (0,1)$.

2.24. Lemma. *Let $a \in \mathbb{R}^n$, $r > 0$, and let $b \in \mathbb{R}^n$, $u > 0$, be such that $B^n(a,r) = Q(b,u)$. If f is the inversion in $S^{n-1}(a,r)$, then*

$$f = t_b^{-1} \circ f_1 \circ t_b \,,$$

where t_b is the spherical isometry defined in (2.22) *and f_1 is the inversion in $S^{n-1}(u/\sqrt{1-u^2}\,) = \partial Q(0,u)$.*

3. Hyperbolic geometry

Hyperbolic geometry can be developed in the context of two spaces or, as they are sometimes called, models. These two models of the hyperbolic space are the unit ball $\mathbf{B}^n$ and the Poincaré half–space

$$\mathbf{H}^n = \mathbb{R}^n_+ = \{\, (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0 \,\} \,.$$

These two models can be equipped with a hyperbolic metric ρ that is unique up to a multiplicative constant in either model. In either model the metric is normalized (by giving the element of length of the metric) in such a way that for all $x, y \in \mathbf{B}^n$

$$\rho_{\mathbf{H}^n}\big(h(x), h(y)\big) = \rho_{\mathbf{B}^n}(x,y)$$

whenever $h \in \mathcal{GM}$ and $h\mathbf{B}^n = \mathbf{H}^n$. Therefore both models are conformally compatible in the sense that the two metric spaces $(\mathbf{B}^n, \rho)$ and $(\mathbf{H}^n, \rho)$ can be identified. This compatibility is very convenient in computations because we may do a computation in that model in which it is easier, without loss of generality. In what follows we shall use the symbols $\mathbb{R}^n_+$ and $\mathbf{H}^n$ interchangeably.

For $A \subset \mathbb{R}^n$ let $A_+ = \{ x \in A : x_n > 0 \}$. We define a weight function $w\colon \mathbb{R}^n_+ \to \mathbb{R}_+ = \{ x \in \mathbb{R} : x > 0 \}$ by

$$w(x) = \frac{1}{x_n} \, , \; x = (x_1, \dots, x_n) \in \mathbb{R}^n_+ \, . \tag{3.1}$$

If $\gamma\colon [0,1) \to \mathbb{R}^n_+$ is a continuous mapping such that $\gamma[0,1)$ is a rectifiable curve with length $s = \ell(\gamma)$, then γ has a normal representation $\gamma^0\colon [0,s) \to \mathbb{R}^n_+$ parametrized by arc length (see J. Väisälä [V, p. 5]). The *hyperbolic length* of $\gamma[0,1)$ is defined by

$$\ell_h(\gamma[0,1)) = \int_0^s |(\gamma^0)'(t)| \; w(\gamma^0(t))dt = \int_\gamma \frac{|dx|}{x_n} \, . \tag{3.2}$$

If $A \subset \mathbb{R}^n_+$ is a (Lebesgue) measurable set we define the *hyperbolic volume* of A by

$$m_h(A) = \int_A w(x)^n \; dm(x) \, , \tag{3.3}$$

where m stands for the n–dimensional Lebesgue measure and w is as in (3.1). If $a, b \in \mathbb{R}^n_+$, then the *hyperbolic distance* between a and b is defined by

$$\rho(a,b) = \inf_{\alpha \in \Gamma_{ab}} \ell_h(\alpha) = \inf_{\alpha \in \Gamma_{ab}} \int_\alpha \frac{|dx|}{x_n} \, , \tag{3.4}$$

where Γ_{ab} stands for the collection of all rectifiable curves in $\mathbb{R}^n_+$ joining a and b. Sometimes the more complete notation $\rho_{\mathbb{R}^n_+}(a,b)$ or $\rho_{\mathbf{H}^n}(a,b)$ will be employed.

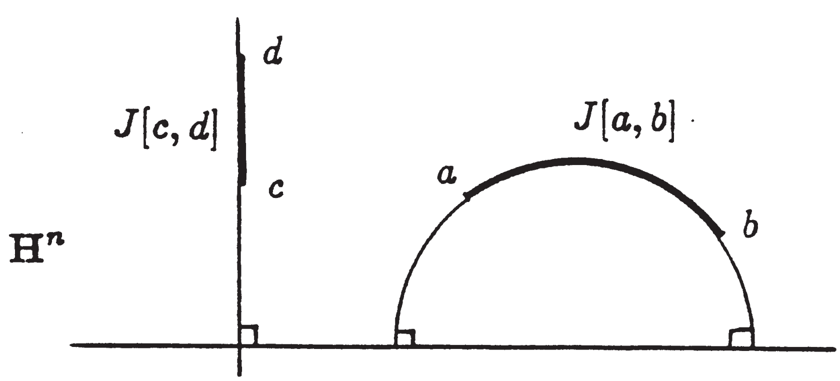

FIGURE 5. Some geodesics of $\mathbf{H}^n = \mathbb{R}^n_+$.

The infimum in (3.4) is in fact attained: for given $a, b \in \mathbb{R}^n_+$ there exists a circular arc L perpendicular to $\partial \mathbb{R}^n_+$ such that the closed subarc $J[a,b]$ of L with end points a and b satisfies

$$\rho(a,b) = \ell_h(J[a,b]) = \int_{J[a,b]} \frac{|dx|}{x_n} \ . \tag{3.5}$$

If a and b are located on a normal of $\partial\mathbb{R}^n_+$, then $J[a,b] = [a,b] = \{\,(1-t)a+tb : 0 \le t \le 1\}$ (cf. [BE, p. 134]). Because of the (hyperbolic) length–minimizing property (3.5), the arc $J[a,b]$ will be called the *geodesic segment* joining a and b.

Knowing the geodesics, we calculate the hyperbolic distance in two special cases. First, for $r, s > 0$ we obtain

$$\rho(re_n, se_n) = \Big| \int_s^r \frac{dt}{t} \Big| = \Big| \log \frac{r}{s} \Big| \ . \tag{3.6}$$

Second, if $\varphi \in (0, \frac{1}{2}\pi)$ we denote $u_\varphi = (\cos\varphi)e_1 + (\sin\varphi)e_n$ and calculate

$$\rho(e_n, u_\varphi) = \int_{J[u_\varphi, e_n]} \frac{d\alpha}{\sin\alpha} = \int_{\varphi}^{\pi/2} \frac{d\alpha}{\sin\alpha} = \log\cot\tfrac{1}{2}\varphi \ . \tag{3.7}$$

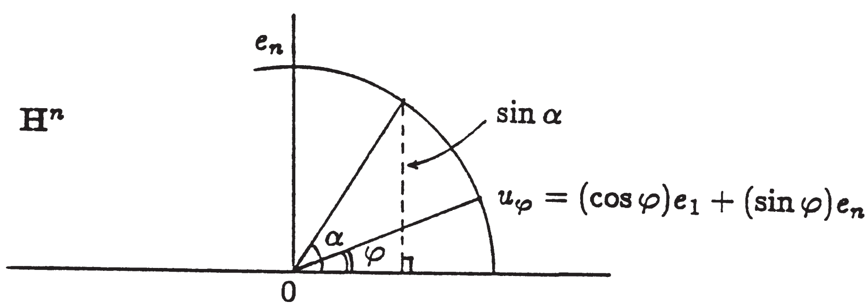

FIGURE 6. The points u_φ and e_n .

We shall often make use of the hyperbolic functions $\operatorname{sh} x = \sinh x$, $\operatorname{ch} x = \cosh x$, $\operatorname{th} x = \tanh x$, $\operatorname{cth} x = \coth x$ and their inverse functions. The above formulae (3.6) and (3.7) are special cases of the general formula (see [BE, p. 35])

$$\operatorname{ch}\rho(x,y) = 1 + \frac{|x-y|^2}{2x_n y_n} \ , \quad x, y \in \mathbf{H}^n = \mathbb{R}^n_+ \ . \tag{3.8}$$

Note that by this formula the hyperbolic distance $\rho(x,y)$ is completely determined once the euclidean distances $x_n = d(x, \partial\mathbf{H}^n)$, $y_n = d(y, \partial\mathbf{H}^n)$, and $|x-y|$ are known. In passing we note that if $f_2 \in \mathcal{GM}(\mathbf{H}^n)$ is as defined in Remark 2.3(4), then $\rho(x,y) = \rho(f_2(x), f_2(y))$ for all $x, y \in \mathbf{H}^n$. For another formulation of (3.8) let $z, w \in \mathbf{H}^n$, let L be an arc of a circle perpendicular to $\partial\mathbf{H}^n$ with $z, w \in L$ and let $\{z_*, w_*\} = L \cap \partial\mathbf{H}^n$, the points being labelled so that z_*, z, w, w_* occur in this order on L. Then (cf. [BE, p. 133, (7.26)])

$$\rho(z,w) = \log|\, z_*, z, w, w_* \,| \ . \tag{3.9}$$

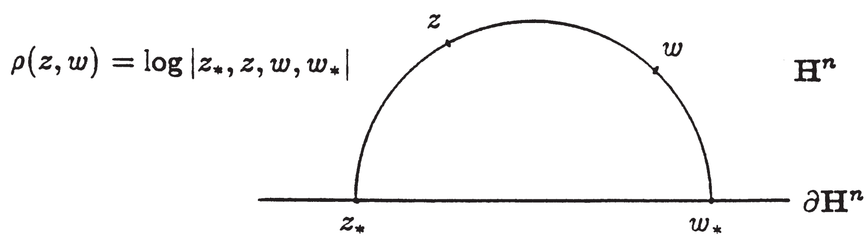

FIGURE 7. The quadruple z_*, z, w, w_*.

Note that (3.6) is a special case of (3.9) when $z_* = 0$ and $w_* = \infty$ because $|\,0, z, w, \infty\,| = |w|/|z|$ for $z, w \in \mathbf{H}^n$. The invariance of ρ is apparent by (3.9) and (2.18): If f in $\mathcal{GM}(\mathbf{H}^n)$, then for all $x, y \in \mathbf{H}^n$

$$\rho(x,y) = \rho(f(x), f(y)) \,. \tag{3.10}$$

For $a \in \mathbf{H}^n$ and $M > 0$ the *hyperbolic ball* $\{\, x \in \mathbf{H}^n : \rho(a,x) < M \,\}$ is denoted by $D(a, M)$. It is well known that $D(a, M) = B^n(z, r)$ for some z and r (this also follows from (3.10)!). This fact together with the observation that $\lambda t e_n$, $(t/\lambda)e_n \in \partial D(te_n, M)$, $\lambda = e^M$ (cf. (3.6)), yields

$$\begin{cases} D(te_n, M) = B^n\big((t \operatorname{ch} M)e_n, t \operatorname{sh} M\big) \,, \\ B^n(te_n, rt) \subset D(te_n, M) \subset B^n(te_n, Rt) \,, \\ r = 1 - e^{-M} \,, \quad R = e^M - 1 \,. \end{cases} \tag{3.11}$$

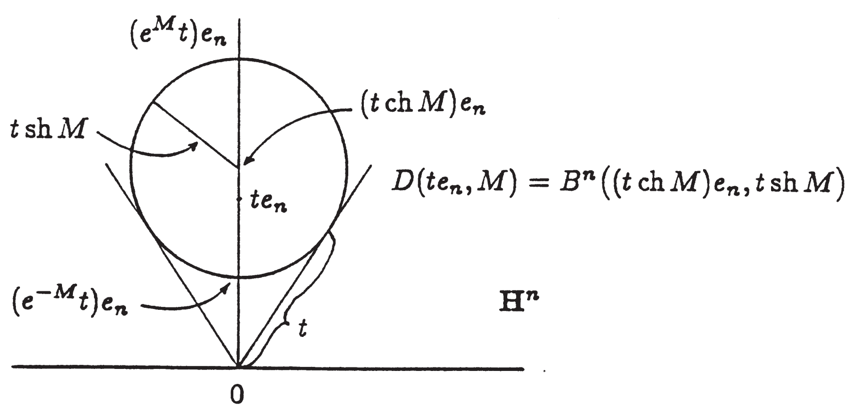

FIGURE 8. The hyperbolic ball $D(te_n, M)$ as a euclidean ball.

A counterpart of (3.8) for $\mathbf{B}^n$ is

$$\operatorname{sh}^2\big(\tfrac{1}{2}\rho(x,y)\big) = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)} \,, \;\; x, y \in \mathbf{B}^n \,, \tag{3.12}$$

(cf. [BE, p. 40]). As in the case of $\mathbf{H}^n$, we see by (3.12) that the hyperbolic distance $\rho(x, y)$ between x and y is completely determined by the euclidean quantities $|x-y|$, $d(x,\partial\mathbf{B}^n)$, $d(y,\partial\mathbf{B}^n)$. Finally, we have also

$$\rho(x,y) = \log|\, x_*, x, y, y_*| \ , \tag{3.13}$$

where x_*, y_* are defined as in (3.9): If L is the circle orthogonal to S^{n-1} with $x,y \in L$, then $\{x_*,\, y_*\} = L \cap S^{n-1}$, the points being labelled so that x_*, x, y, y_* occur in this order on L. It follows from (3.13) and (2.18) that

$$\rho(x,y) = \rho(h(x), h(y)) \tag{3.14}$$

for all $x, y \in \mathbf{B}^n$ whenever h is in $\mathcal{GM}(\mathbf{B}^n)$. Finally, in view of (2.18), (3.9), and (3.13) we have

$$\rho_{\mathbf{B}^n}(x,y) = \rho_{\mathbf{H}^n}(g(x), g(y)) \ , \quad x, y \in \mathbf{B}^n \ , \tag{3.15}$$

whenever g is a Möbius transformation with $g\mathbf{B}^n = \mathbf{H}^n$.

It is well known that the balls $D(z,M)$ of $(\mathbf{B}^n, \rho)$ are balls in the euclidean geometry as well, i.e. $D(z,M) = B^n(y,r)$ for some $y \in \mathbf{B}^n$ and $r > 0$. Making use of this fact, we shall find y and r. Let L_z be a euclidean line through 0 and z and $\{z_1, z_2\} = L_z \cap \partial D(z,M)$, $|z_1| \le |z_2|$. We may assume that $z \neq 0$ since with obvious changes the following argument works for $z = 0$ as well. Let $e = z/|z|$ and $z_1 = se$, $z_2 = ue$, $u \in (0,1)$, $s \in (-u,u)$. Then it follows that

$$\begin{aligned} \rho(z_1, z) &= \log\Big(\frac{1+|z|}{1-|z|} \cdot \frac{1-s}{1+s}\Big) = M \ , \\ \rho(z_2, z) &= \log\Big(\frac{1+u}{1-u} \cdot \frac{1-|z|}{1+|z|}\Big) = M \ . \end{aligned}$$

Solving these for s and u and using the fact that

$$D(z,M) = B^n\big(\tfrac{1}{2}(z_1+z_2),\ \tfrac{1}{2}|u-s|\big)$$

one obtains the following formulae:

$$\begin{cases} D(x,M) = B^n(y,r) \\[2ex] y = \dfrac{x(1-t^2)}{1-|x|^2t^2} \ , \quad r = \dfrac{(1-|x|^2)t}{1-|x|^2t^2} \ , \quad t = \operatorname{th}\tfrac{1}{2}M \ , \end{cases} \tag{3.16}$$

and

$$\begin{cases} B^n\big(x,\ a(1-|x|)\big) \subset D(x,M) \subset B^n\big(x,\ A(1-|x|)\big) \ , \\[2ex] a = \dfrac{t(1+|x|)}{1+|x|t} \ , \quad A = \dfrac{t(1+|x|)}{1-|x|t} \ , \quad t = \operatorname{th}\tfrac{1}{2}M \ . \end{cases} \tag{3.17}$$

We shall often need a special case of (3.16):

$$D(0,M) = B^n(\operatorname{th}\frac{1}{2}M) \ . \tag{3.18}$$

A standard application of formula (3.18) is the following observation. Let T_x be in $\mathcal{M}(\mathbf{B}^n)$ as defined in 2.19 with $T_x(x)=0$. Fix $x,y\in\mathbf{B}^n$ and $z\in J[x,y]$ with $\rho(z,x)=\rho(z,y)=\frac{1}{2}\rho(x,y)$. Then $T_z(x)=-T_z(y)$ and (3.18) yields

$$\text{(3.19)}\qquad \begin{cases} |T_x(y)| = \operatorname{th}\frac{1}{2}\rho(x,y)\ , \\ |T_z(x)| = \operatorname{th}\frac{1}{4}\rho(x,y)\ . \end{cases}$$

For an open set D in $\mathbb{R}^n$, $D\neq\mathbb{R}^n$, define $d(z)=d(z,\partial D)$ for $z\in D$ and

$$\text{(3.20)}\qquad j_D(x,y)=\log\Big(1+\frac{|x-y|}{\min\{d(x),\,d(y)\}}\Big)$$

for $x,y\in D$. Then it is well-known that j_D is a metric (see, e.g. [S]).

3.21. Lemma. *The following inequalities*

$$(1)\ \ j_D(x,y)\ \ \geq\ \ \Big|\log\frac{d(x)}{d(y)}\Big|\ ,$$

$$(2)\ \ j_D(x,y)\ \ \leq\ \ \Big|\log\frac{d(x)}{d(y)}\Big|+\log\Big(1+\frac{|x-y|}{d(x)}\Big)\leq 2\,j_D(x,y)$$

$$(3)\ \ j_D(x,y)\ \ \geq\ \ \Big|\log\frac{|x-z|}{|y-z|}\Big|$$

hold for all $x,y\in D, z\in\partial D$.

In the next lemma we show that j_D yields simple two–sided estimates for ρ_D both when $D=\mathbf{B}^n$ and when $D=\mathbf{H}^n$.

3.22. Lemma. (1) $j_{\mathbf{B}^n}(x,y)\leq\rho_{\mathbf{B}^n}(x,y)\leq 4\,j_{\mathbf{B}^n}(x,y)$ *for* $x,y\in\mathbf{B}^n$.
(2) $j_{\mathbf{H}^n}(x,y)\leq\rho_{\mathbf{H}^n}(x,y)\leq 2\,j_{\mathbf{H}^n}(x,y)$ *for* $x,y\in\mathbf{H}^n$.

4. Quasihyperbolic geometry

In an arbitrary proper subdomain D of $\mathbb{R}^n$ one can define a metric, the quasihyperbolic metric of D, which shares some properties of the hyperbolic metric of $\mathbf{B}^n$ or $\mathbf{H}^n$. We shall now give the definition of the quasihyperbolic metric and state without proof some of its basic properties which we require later on. The quasihyperbolic metric has been systematically developed and applied by F. W. Gehring and his collaborators.

Throughout this section D will denote a proper subdomain of $\mathbb{R}^n$. In D we define a weight function $w\colon D\to\mathbb{R}_+$ by

$$\text{(4.1)}\qquad w(x)=\frac{1}{d(x,\partial D)}\ ;\ \ x\in D\ .$$

Using this weight function one defines the *quasihyperbolic length* $\ell_q(\gamma)=\ell_q^D(\gamma)$ of a rectifiable curve γ by a formula similar to (3.2). The *quasihyperbolic distance* between x and y in D is defined by

$$\text{(4.2)}\qquad k_D(x,y)=\inf_{\alpha\in\Gamma_{xy}}\ell_q^D(\alpha)=\inf_{\alpha\in\Gamma_{xy}}\int_\alpha w(x)|dx|\ ,$$

where Γ_{xy} is as in (3.4). It is clear that k_D is a metric on D. It follows from (4.2) that k_D is invariant under translations, stretchings, and orthogonal mappings. (As in (3.3) one can define the *quasihyperbolic volume* of a (Lebesgue) measurable set $A \subset D$, but we shall not make use of this notion.) Given $x, y \in D$ there exists a *geodesic segment* $J_D[x, y]$ of the metric k_D joining x and y (cf. [GO]). However, very little is known about the structure of such geodesic segments $J_D[x, y]$ when D is given. For some elementary domains, the geodesics were recently studied by H. Lindén [L].

4.3. Remarks. Clearly, $k_{\mathbf{H}^n} = \rho_{\mathbf{H}^n}$, and we see easily that $\rho_{\mathbf{B}^n} \le 2\, k_{\mathbf{B}^n} \le 2\, \rho_{\mathbf{B}^n}$ (cf. (4.1)). Hence, the geodesics of $(\mathbf{H}^n, k_{\mathbf{H}^n})$ are those of $(\mathbf{H}^n, \rho_{\mathbf{H}^n})$, but it is a difficult task to find the geodesics of k_D when D is given. The following monotone property of k_D is clear: if D and D' are domains with $D' \subset D$ and $x, y \in D'$, then $k_{D'}(x, y) \ge k_D(x, y)$.

In order to find some estimates for $k_D(x, y)$ we shall employ, as in the case of $\mathbf{H}^n$ and $\mathbf{B}^n$, the metric j_D defined in (3.20). The metric j_D is indeed a natural choice for such a comparison function since both k_D and j_D are invariant under translations, stretchings and orthogonal mappings. A useful inequality is ([GP, Lemma 2.1])

$$k_D(x, y) \ge j_D(x, y)\ ;\ \ x,\, y \in D\ . \tag{4.4}$$

In combination with 3.22, (4.4) yields

$$k_D(x, y) \ge \left| \log \frac{d(x)}{d(y)} \right|,\ \ d(z) = d(z, \partial D)\ . \tag{4.5}$$

For easy reference we record Bernoulli's inequality

$$\log(1 + as) \le a \log(1 + s)\ ;\ a \ge 1\,,\ \ s > 0\ . \tag{4.6}$$

4.7. Lemma. (1) *If $x \in D$, $y \in B_x = B^n(x, d(x))$, then*

$$k_D(x, y) \le \log\Big(1 + \frac{|x - y|}{d(x) - |x - y|}\Big)\ .$$

(2) *If $s \in (0, 1)$ and $|x - y| \le s\, d(x)$, then*

$$k_D(x, y) \le \frac{1}{1 - s}\, j_D(x, y)\ .$$

5. Modulus and capacity

For the definition and basic properties of the modulus we refer the reader to [V1]. The main sources for this section are [V1], [Vu3], [AVV2].

One of the main reasons why the modulus of a curve family is studied is that we have a simple rule of transformation for the modulus of a curve family under quasiconformal mappings. Further, we would like to use modulus as an instrument so as to gain insight about "the geometry". Roughly speaking we can say that the modulus of the family of all curves joining two connected non-intersecting continua $E, F \subset \mathbb{R}^n$ behaves like $\min\{d(E), d(F)\}/d(E, F)$, where

d stands for the euclidean diameter. A long series of estimates is needed to reach this conclusion and its variants and some of these estimates are given in this and the following section. Some of these variants involve hyperbolic or quasihyperbolic metric. In this fashion we step by step approach our goal, the study of how quasiconformal mappings between metric spaces deform distances.

5.1. Lemma. *The p–modulus M_p is an outer measure in the space of all curve families in $\mathbb{R}^n$. That is,*

$$(1)\qquad \mathsf{M}_p(\emptyset) = 0\,,$$
$$(2)\qquad \Gamma_1 \subset \Gamma_2 \text{ implies } \mathsf{M}_p(\Gamma_1) \le \mathsf{M}_p(\Gamma_2)\,,$$
$$(3)\qquad \mathsf{M}_p\Big(\bigcup_{i=1}^{\infty} \Gamma_i\Big) \le \sum_{i=1}^{\infty} \mathsf{M}_p(\Gamma_i)\,.$$

Let Γ_1 and Γ_2 be curve families in $\mathbb{R}^n$. We say that Γ_2 is *minorized* by Γ_1 and write $\Gamma_2 > \Gamma_1$ if every $\gamma \in \Gamma_2$ has a subcurve belonging to Γ_1.

5.2. Lemma. $\Gamma_1 < \Gamma_2$ *implies* $\mathsf{M}_p(\Gamma_1) \ge \mathsf{M}_p(\Gamma_2)$.

5.3. Remark. The family of all paths joining E and F in G is denoted by $\Delta(E,F;G)$ see [Vu3, p.51]. If $G = \mathbb{R}^n$ or $\overline{\mathbb{R}}^n$ we often denote $\Delta(E,F;G)$ by $\Delta(E,F)$. Curve families of this form are the most important for what follows. The following subadditivity property is useful. If $E = \bigcup_{j=1}^{\infty} E_j$ and $c_E(F) = \mathsf{M}_p\big(\Delta(E,F)\big) = c_F(E)$, then $c_F(E) \le \sum c_F(E_j)$, see 5.1(3). More precisely if $G \subset \overline{\mathbb{R}}^n$ is a domain and $F \subset G$ is fixed, then $c_F^G(E) = \mathsf{M}_p\big(\Delta(E,F;G)\big)$ is an outer measure defined for $E \subset G$. In a sense which will be made precise later on, $c_E(F)$ describes the mutual size and location of E and F. Assume now that D is an open set in $\overline{\mathbb{R}}^n$ and that $F \subset D$. It follows from 5.1(2) that

$$\mathsf{M}_p\big(\Delta(F,\partial D; D\setminus F)\big) \le \mathsf{M}_p\big(\Delta(F,\partial D; D)\big) \le \mathsf{M}_p\big(\Delta(F,\partial D)\big)\,.$$

On the other hand, because $\Delta(F,\partial D;D) < \Delta(F,\partial D)$ and $\Delta(F,\partial D; D\setminus F) < \Delta(F,\partial D;D)$, 5.2 yields

$$(5.4)\qquad \mathsf{M}_p\big(\Delta(F,\partial D)\big) = \mathsf{M}_p\big(\Delta(F,\partial D;D)\big) = \mathsf{M}_p\big(\Delta(F,\partial D; D\setminus F)\big)\,.$$

5.5. Lemma. *Let D and D' be domains in $\overline{\mathbb{R}}^n$ and let $f\colon D \to D'$ be a conformal mapping. Then $\mathsf{M}(f\Gamma) = \mathsf{M}(\Gamma)$ for each curve family Γ in D where $f\Gamma = \{\, f\circ\gamma : \gamma \in \Gamma\,\}$.*

5.6. Lemma. *Let $p > 1$ and let E and F be subsets of $\mathbb{R}^n_+$. Then*

$$\mathsf{M}_p\big(\Delta(E,F;\mathbb{R}^n_+)\big) \ge \frac{1}{2}\,\mathsf{M}_p\big(\Delta(E,F)\big)\,.$$

5.7. Corollary. *Let E and F be sets in $\overline{\mathbb{R}}^n$ with $q(\,\overline{E},\overline{F}\,) \ge a > 0$. Then $\mathsf{M}\big(\Delta(E,F)\big) \le c(n,a) < \infty$.*

5.8. Lemma. *(1) Let $0 < a < b$ and let E, F be sets in $\mathbb{R}^n$ with*

$$E \cap S^{n-1}(t) \ne \emptyset \ne F \cap S^{n-1}(t)$$

for $t \in (a,b)$. *Then*

$$\mathsf{M}\big(\Delta(\,E,F;B^n(b)\setminus B^n(a)\,)\big) \ge c_n \log\frac{b}{a}\,.$$

Equality holds if $E=(ae_1,be_1)$, $F=(-be_1,-ae_1)$. *Here* $c_n>0$ *depends only on* n *(see* [V1, (10.11),(10.4)]*).*

(2) Let $0<a<b$. *Then*

$$\mathsf{M}\big(\Delta(\,S^{n-1}(a),S^{n-1}(a);B^n(b)\setminus B^n(a)\,)\big) = \omega_{n-1}(\log(b/a))^{1-n}\,,$$

where ω_{n-1} *is the* $(n-1)$*-dimensional surface area of* S^{n-1}.

5.9. Corollary. *If* E *and* F *are non–degenerate continua with* $0\in E\cap F$ *then* $\mathsf{M}\big(\Delta(E,F)\big)=\infty$.

Proof. Apply 5.8 with a fixed b such that $S^{n-1}(b)\cap E\ne\emptyset\ne S^{n-1}(b)\cap F$ and let $a\to 0$. □

5.10. Canonical ring domains. A domain (open, connected set) D in $\mathbb{R}^n$ is called a *ring domain* or, briefly, a *ring*, if $\mathbb{R}^n\setminus D$ consists of two components C_0 and C_1. Sometimes we denote such a ring by $R(C_0,C_1)$. In our study two canonical ring domains will be of particular importance. These are the *Grötzsch ring* $R_{G,n}(s)$, $s>1$, and the *Teichmüller ring* $R_{T,n}(t)$, $t>0$, defined by

$$\begin{cases} R_{G,n}(s) &= R(\overline{\mathbf{B}^n},[se_1,\infty]), \quad s>1,\\ R_{T,n}(t) &= R([-e_1,0],[te_1,\infty]), \quad t>0.\end{cases} \tag{5.11}$$

Sometimes we also use the bounded Grötzsch ring $R(\mathbb{R}^n\setminus\mathbf{B}^n,[0,re_1])$. An important conformal invariant associated with a ring is the modulus of the family of curves joining its complementary components. In the case of Grötzsch ring $R_{G,n}(s)$ and Teichmüller ring $R_{T,n}(t)$ the modulus is denoted by $\gamma_n(s)$ and $\tau_n(t)$ respectively. It is a well-known basic fact that $\gamma_n:(1,\infty)\to(0,\infty)$ is a decreasing homeomorphism and that for all $s>1$

$$\gamma_n(s) = 2^{n-1}\tau_n(s^2-1)\,. \tag{5.12}$$

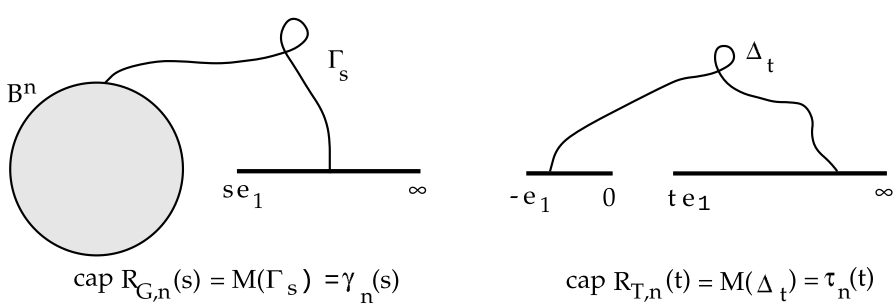

FIGURE 9. The Grötzsch and Teichmüller rings.

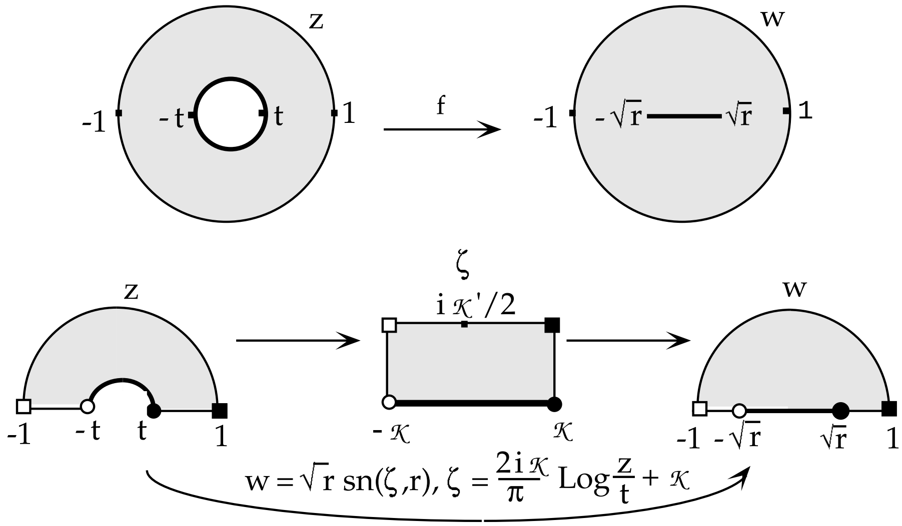

FIGURE 10. Conformal map of an annulus onto a disk minus a symmetric slit.

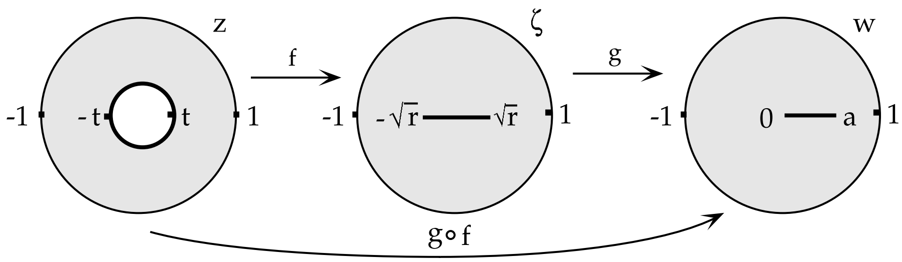

FIGURE 11. Conformal map of an annulus onto a bounded Grötzsch ring.

5.13. Elliptic integrals and $\gamma_2(s)$. In the plane every ring domain can be conformally mapped onto an annulus $\{z \in \mathbb{C}\colon 1 < |z| < M\}$ for some M. For the Grötzsch ring this conformal mapping is given by the elliptic sn-function [AVV2]. For more information on the involved special functions see [QV].

As shown in [LV, II.2]

$$\gamma_2(s) = 2\pi/\mu(1/s) \tag{5.14}$$

for $s > 1$ where

$$\mu(r) = \frac{\pi}{2}\,\frac{K(\sqrt{1-r^2}\,)}{K(r)}\ ,\ K(r) = \int_0^1 [(1-x^2)(1-r^2x^2)]^{-1/2}dx$$

for $0 < r < 1$. The function $K(r)$ is called a complete elliptic integral of the first kind and its values can be found in tables.

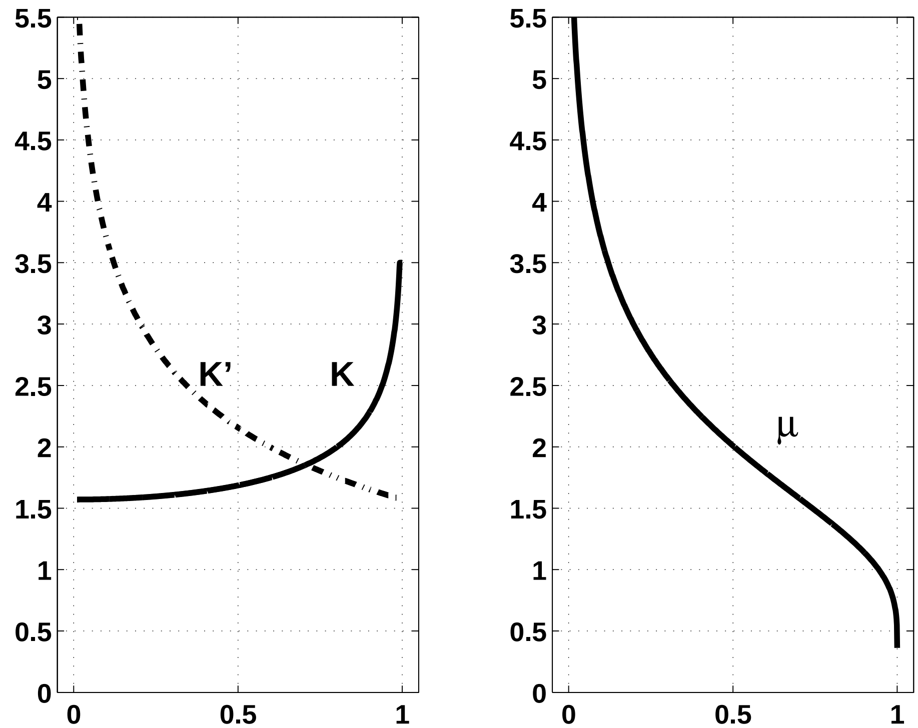

FIGURE 12. The functions $K(r)$ and $\mu(r)$.

The modulus $\mu(r)$ satisfies the following three functional identities

$$\begin{cases} \mu(r)\mu\big(\sqrt{1-r^2}\big) = \frac{1}{4}\pi^2 \ , \\ \mu(r)\mu\Big(\frac{1-r}{1+r}\Big) = \frac{1}{2}\pi^2 \ , \\ \mu(r) = 2\mu\Big(\frac{2\sqrt{r}}{1+r}\Big) \ . \end{cases} \tag{5.15}$$

From (5.15) one can derive several estimates for $\mu(r)$ [LV, p. 62]. By [LV, p. 62] the following inequalities hold

$$\log\frac{1}{r} < \log\frac{1+3\sqrt{1-r^2}}{r} < \mu(r) < \log\frac{4}{r} \tag{5.16}$$

for $0 < r < 1$. From (5.16) it follows that $\lim_{r\to 0+}\mu(r) = \infty$ whence, by virtue of the functional identities (5.15), $\lim_{r\to 1-}\mu(r) = 0$. Therefore, $\mu\colon (0,1)\to(0,\infty)$ is a decreasing homeomorphism. For the sake of completeness we set $\mu(0)=\infty$ and $\mu(1) = 0$. By (5.14) and (5.15) we obtain

$$\gamma_2(s) = \frac{4}{\pi}\,\mu\Big(\frac{s-1}{s+1}\Big)\ ,\ s > 1\ . \tag{5.17}$$

5.18. Exercise. In the study of distortion theory of quasiconformal mappings in Section 7 below the following special function will be useful

$$\varphi_{K,n}(r) = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/r))}$$

for $0 < r < 1$, $K > 0$. (Note: [Vu1, Lemma 7.20] shows that γ_n is strictly decreasing and hence that γ_n^{-1} exists.) Show that $\varphi_{AB,n}(r) = \varphi_{A,n}(\varphi_{B,n}(r))$ and $\varphi_{A,n}^{-1}(r) = \varphi_{1/A,n}(r)$ and that

$$\varphi_{K,2}(r) = \varphi_K(r) = \mu^{-1}\big(\tfrac{1}{K}\mu(r)\big) .$$

Verify also that

$$(1) \qquad \varphi_2(r) = \tfrac{2\sqrt{r}}{1+r}$$

$$(2) \quad \varphi_K(r)^2 + \varphi_{1/K}\big(\sqrt{1-r^2}\,\big)^2 = 1 .$$

Exploiting (1) and (2) find $\varphi_{1/2}(r)$. Show also that

$$(3) \quad \varphi_{1/K}\left(\tfrac{1-r}{1+r}\right) = \tfrac{1-\varphi_K(r)}{1+\varphi_K(r)} ,$$

$$(4) \quad \varphi_K\left(\tfrac{2\sqrt{r}}{1+r}\right) = \tfrac{2\sqrt{\varphi_K(r)}}{1+\varphi_K(r)} .$$

Lemma 7.22 in [Vu3] yields the inequalities

$$\omega_{n-1}\left(\log \lambda_n s\right)^{1-n} \le \gamma_n(s) \le \omega_{n-1}(\log s)^{1-n} , \tag{5.19}$$

$$\omega_{n-1}\left(\log(\lambda_n^2 s)\right)^{1-n} \le \tau_n(s-1) \le \omega_{n-1}(\log s)^{1-n} , \tag{5.20}$$

for $s > 1$.

5.21. Theorem. *The function $g_n(t) = (\omega_{n-1}/\gamma_n(t))^{1/(n-1)} - \log t$ is an increasing function on $(1,\infty)$ with $\lim_{t\to\infty} g_n(t) = \log\lambda_n$ where $\lambda_n \in [4, 2e^{n-1}), \lambda_2 = 4$, is so-called Grötzsch ring constant.*

5.22. Theorem. *For $s \in (1,\infty)$ and $n \ge 2$*

(1) $\gamma_n(s) \le \omega_{n-1}\mu(1/s)^{1-n} < \omega_{n-1}\big(\log(s + 3\sqrt{s^2-1}\,)\big)^{1-n}$,

(2) $2^{n-1}c_n \log\Big(\dfrac{s+1}{s-1}\Big) \le \gamma_n(s) \le 2^{n-1}c_n\,\mu\Big(\dfrac{s-1}{s+1}\Big) < 2^{n-1}c_n \log\Big(4\,\dfrac{s+1}{s-1}\Big)$.

Moreover, if $s \in (0,\infty)$ and $a = 1 + 2(1+\sqrt{1+s}\,)/s$, then

(3) $c_n \log a \le \tau_n(s) \le c_n\mu(1/a) < c_n \log(4a)$

and $(1 + 1/\sqrt{s}\,)^2 \le a \le (1 + 2/\sqrt{s}\,)^2$ hold true. Furthermore, when $n = 2$, the first inequality in (1), the second inequality in (2), and the second inequality in (3) hold as identities.

5.23. Hyperbolic metric and capacity. As in Section 3 we let $J[x,y]$ denote the geodesic segment of the hyperbolic metric joining x to y, $x, y \in \mathbf{B}^n$. It is clear by conformal invariance that

$$\mathrm{cap}(\mathbf{B}^n, J[x,y]) = \mathrm{cap}(\mathbf{B}^n, T_x J[x,y])$$

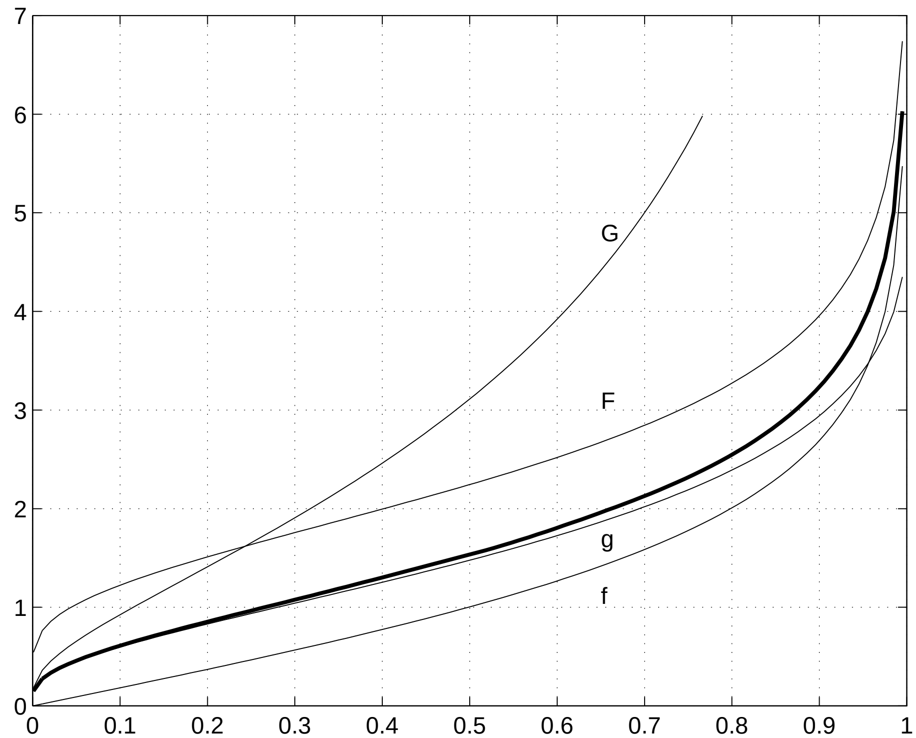

FIGURE 13. Bounds for γ_3.

where T_x is as defined in 2.19. We obtain by (3.19) and [Vu1, (7.25)]

$$\text{(5.24)} \quad \operatorname{cap}(\mathbf{B}^n, J[x,y]) = \gamma_n\Big(\frac{1}{\operatorname{th}\frac{1}{2}\rho(x,y)}\Big) \le \omega_{n-1}\big(-\log \operatorname{th}\frac{1}{4}\rho(x,y)\big)^{1-n} .$$

Next by (5.24), 5.22(2), and (5.16) we get

$$\begin{aligned} 2^{n-1}c_n\,\rho(x,y) &\le \operatorname{cap}(\mathbf{B}^n, J[x,y]) \le 2^{n-1}c_n\,\mu(e^{-\rho(x,y)}) \\ \text{(5.25)} \qquad &< 2^{n-1}c_n(\rho(x,y)+\log 4) . \end{aligned}$$

For large values of $\rho(x,y)$ (5.25) is quite accurate. For small $\rho(x,y)$ one obtains better inequalities than (5.25) by combining 5.22(1) and (5.24).

It is left as an easy exercise for the reader to derive from (5.19) the following inequality

$$\text{(5.26)} \qquad t^\alpha/\lambda_n \le \gamma_n^{-1}(K\gamma_n(t)) \le \lambda_n^\alpha\, t^\alpha$$

for all $t > 1$ and $K > 0$, where $\alpha = K^{1/(1-n)}$. From (5.26) it follows immediately that

$$\text{(5.27)} \qquad r^\alpha {\lambda_n}^{-\alpha} \le \varphi_{K,n}(r) \le \lambda_n r^\alpha$$

holds for all $K > 0$ and $r \in (0,1)$. For $K \ge 1$ this inequality can be refined if we use Theorem 5.22 (1).

5.28. Theorem. *For $n \ge 2$, $K \ge 1$, and $0 \le r \le 1$*

$$(1) \quad \varphi_K(r) \le \lambda_n^{1-\alpha}\, r^{\alpha}\,, \quad \alpha = K^{1/(1-n)}\,,$$
$$(2) \quad \varphi_{1/K}(r) \ge \lambda_n^{1-\beta}\, r^{\beta}\,, \quad \beta = K^{1/(n-1)}\,.$$

A compact set $E \subset \mathbf{B}^n$ is said to be of capacity zero, denoted $\operatorname{cap} E = 0$, if $\mathsf{M}(\Delta(E, S^{n-1}(2))) = 0$. A compact set $E \subset \overline{\mathbb{R}}^n$ is said to be of capacity zero, if it can be mapped by a Möbius transformation onto a set $E_1 \subset \mathbf{B}^n$ of capacity zero. Sets of capacity zero are very small: they have zero Hausdorff dimension, see [Vu3, p.86]. For many purposes they are negligible. The next theorem provides a convenient tool for estimating moduli of curve families in terms is geometric quantities and a set function.

5.29. Theorem. *For $n \ge 2$ there exist positive numbers $d_1, \dots, d_4$ and a set function $c(\cdot)$ in $\overline{\mathbb{R}}^n$ such that*

(1) $c(E) = c(hE)$ *whenever* $h\colon \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ *is a spherical isometry and* $E \subset \overline{\mathbb{R}}^n$.
(2) $c(\emptyset) = 0$, $A \subset B \subset \overline{\mathbb{R}}^n$ *implies* $c(A) \le c(B)$ *and* $c\big(\bigcup_{j=1}^{\infty} E_j\big) \le d_1 \sum_{j=1}^{\infty} c(E_j)$ *if* $E_j \subset \overline{\mathbb{R}}^n$.
(3) *If* $E \subset \overline{\mathbb{R}}^n$ *is compact, then* $c(E) > 0$ *if and only if* $\operatorname{cap} E > 0$. *Moreover* $c(\overline{\mathbb{R}}^n) \le d_2 < \infty$.
(4) $c(E) \ge d_3\, q(E)$ *if* $E \subset \overline{\mathbb{R}}^n$ *is connected and* $E \ne \emptyset$.
(5) $\mathsf{M}\big(\Delta(E,F)\big) \ge d_4 \min\{\, c(E),\, c(F)\,\}$, *if* $E, F \subset \overline{\mathbb{R}}^n$.

Furthermore, for $n \ge 2$ and $t \in (0,1)$ there exists a positive number d_5 such that

(6) $\mathsf{M}\big(\Delta(E,F)\big) \le d_5 \min\{c(E), c(F)\}$ *whenever* $E, F \subset \overline{\mathbb{R}}^n$ *and* $q(E,F) \ge t$.

6. Conformal invariants

In the preceding sections we have studied some properties of the conformal invariant $\mathsf{M}\big(\Delta(E,F;G)\big)$. In this section we shall introduce two other conformal invariants, the modulus metric $\mu_G(x,y)$ and its "dual" quantity $\lambda_G(x,y)$, where G is a domain in $\overline{\mathbb{R}}^n$ and $x, y \in G$. The modulus metric μ_G is functionally related to the hyperbolic metric ρ_G if $G = \mathbf{B}^n$, while in the general case μ_G reflects the "capacitary geometry" of G in a delicate fashion. The dual quantity $\lambda_G(x,y)$ is also functionally related to ρ_G if $G = \mathbf{B}^n$. As shown in [Vu3] for a wide class of domains in $\mathbb{R}^n$, the so–called QED–domains[GM], two–sided estimates for $\lambda_G(x,y)$ in terms of

$$r_G(x,y) = \frac{|x-y|}{\min\{\, d(x,\partial G),\, d(y,\partial G)\,\}}\ .$$

6.1. The conformal invariants λ_G and μ_G. If G is a proper subdomain of $\overline{\mathbb{R}}^n$, then for $x, y \in G$ with $x \ne y$ we define

$$\lambda_G(x,y) = \inf_{C_x, C_y} \mathsf{M}\big(\Delta(C_x, C_y; G)\big) \tag{6.2}$$

where $C_z = \gamma_z[0,1)$ and $\gamma_z\colon [0,1) \to G$ is a curve such that $\gamma_z(0) = z$ and $\gamma_z(t) \to \partial G$ when $t \to 1$, $z = x,\, y$. It follows from 5.5 that λ_G is invariant under conformal mappings of G. That is, $\lambda_{fG}(f(x), f(y)) = \lambda_G(x, y)$, if $f\colon G \to fG$ is conformal and $x, y \in G$ are distinct.

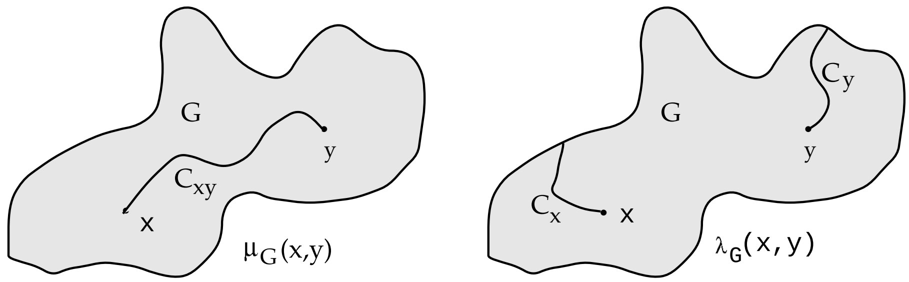

FIGURE 14. The conformal invariants λ_G and μ_G.

6.3. Remark. If $\mathrm{card}(\overline{\mathbb{R}}^n \setminus G) = 1$, then $\lambda_G(x, y) \equiv \infty$ by 5.9. Therefore λ_G is of interest only in case $\mathrm{card}(\overline{\mathbb{R}}^n \setminus G) \geq 2$. For $\mathrm{card}(\overline{\mathbb{R}}^n \setminus G) \geq 2$ and $x, y \in G$, $x \neq y$, there are continua C_x and C_y as in (6.2) with $\overline{C}_x \cap \overline{C}_y = \emptyset$ and thus $\mathsf{M}\big(\Delta(C_x, C_y; G)\big) < \infty$ by 5.7. Thus, if $\mathrm{card}(\overline{\mathbb{R}}^n \setminus G) \geq 2$, we may assume that the infimum in (6.2) is taken over continua C_x and C_y with $\overline{C}_x \cap \overline{C}_y = \emptyset$.

6.4. The extremal problems of Grötzsch and Teichmüller. The Grötzsch and Teichmüller rings arise from extremal problems of the following type, which were first posed for the case of the plane: Among all ring domains which separate two given closed sets E_1 and E_2, $E_1 \cap E_2 = \emptyset$, find one whose module has the greatest value.

Let E_1 be a continuum and E_2 consist of two points not separated by E_1. By the conformal invariance of the modulus one may then suppose that $E_1 = S^1$ and $E_2 = \{0, r\}$, $0 < r < 1$. Then the extremal problem is solved by the bounded Grötzsch ring $R(\mathbb{R}^2 \setminus B^2, [0, r])$. In other words, $\mathrm{cap}(B^2, E) \geq \gamma_2(1/r)$, where $E \subset B^2$ is any continuum joining the points 0 and $r \in \mathbb{R}$. For details we refer the reader to [LV, Ch. II].

The following function is the solution of the generalization of the Teichmüller problem to $\mathbb{R}^n$. For $x \in \mathbb{R}^n \setminus \{0, e_1\}$, $n \geq 2$, define

$$p(x) = \inf_{E,F} \mathsf{M}(\triangle(E, F)), \tag{6.5}$$

where the infimum is taken over all pairs of continua E and F in $\overline{\mathbb{R}}^n$ with $0, e_1 \in E$, $x, \infty \in F$. Teichmüller applied a symmetrization method to prove that for $n = 2$,

$$p(x) \geq p((1 + |x - e_1|)e_1)$$

with equality for $x = (1 + |x - e_1|)e_1$. For more details, see [HV] and [SoV].

For a proper subdomain G of $\overline{\mathbb{R}}^n$ and for all $x, y \in G$ define

$$\mu_G(x, y) = \inf_{C_{xy}} \mathsf{M}\big(\Delta(C_{xy}, \partial G; G)\big) \tag{6.6}$$

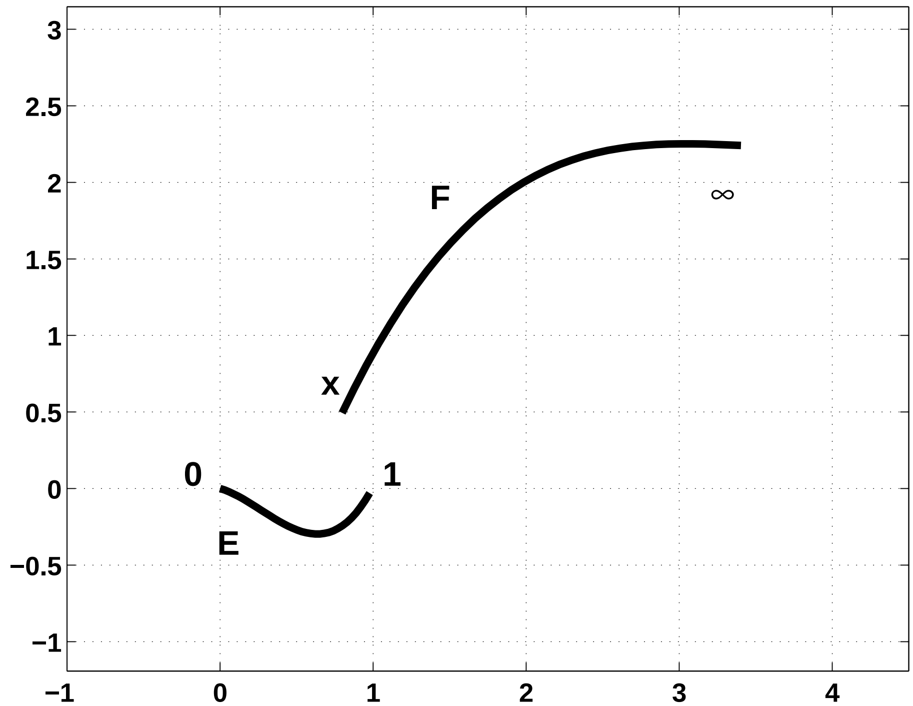

FIGURE 15. The extremal problem of Teichmüller.

where the infimum is taken over all continua C_{xy} such that $C_{xy} = \gamma[0,1]$ and γ is a curve with $\gamma(0) = x$ and $\gamma(1) = y$. It is clear that μ_G is also a conformal invariant in the same sense as λ_G. It is left as an easy exercise for the reader to verify that μ_G is a metric if $\mathrm{cap}\partial G > 0$. [Hint: Apply 5.3 and 5.29.] If $\mathrm{cap}\partial G > 0$, we call μ_G the *modulus metric* or *conformal metric* of G.

6.7. Remark. Let D be a subdomain of G. It follows from 5.3 and (5.4) that $\mu_G(a,b) \le \mu_D(a,b)$ for all $a,b \in D$ and $\lambda_G(a,b) \ge \lambda_D(a,b)$ for all distinct $a,b \in D$. In what follows we are interested only in the non–trivial case $\mathrm{card}(\overline{\mathbb{R}}^n \setminus G) \ge 2$. Moreover, by performing an auxiliary Möbius transformation, we may and shall assume that $\infty \in \overline{\mathbb{R}}^n \setminus G$ throughout this section. Hence G will have at least one finite boundary point.

In a general domain G, the values of $\lambda_G(x,y)$ and $\mu_G(x,y)$ cannot be expressed in terms of well–known simple functions. For $G = \mathbf{B}^n$ they can be given in terms of $\rho(x,y)$ and the capacity of the Teichmüller condenser.

6.8. Theorem. *The following identities hold for all distinct $x, y \in \mathbf{B}^n$:*

$$(1) \qquad \mu_{\mathbf{B}^n}(x,y) = 2^{n-1}\tau\left(\frac{1}{\mathrm{sh}^2\,\frac{1}{2}\rho(x,y)}\right) = \gamma\left(\frac{1}{\mathrm{th}\,\frac{1}{2}\rho(x,y)}\right),$$

$$(2) \qquad \lambda_{\mathbf{B}^n}(x,y) = \frac{1}{2}\tau\left(\mathrm{sh}^2\,\frac{1}{2}\rho(x,y)\right).$$

6.9. Remark. (1) In [Vu3, p. 193] it was stated as an open problem, whether $\lambda_D(x,y)^{1/(1-n)}$ is a metric when $D = \mathbb{R}^n \setminus \{0\}$ and $n = 2$. Subsequently the problem was solved by A. Solynin [So] and J. Jenkins [J] for $n = 2$. J. Ferrand [F] proved that $\lambda_D(x,y)^{1/(1-n)}$ is a metric for all $D \subset R^n, n \geq 2$.

(2) From 5.22(3) we obtain the following inequality for $x, y \in \mathbf{B}^n$ (exercise)

$$\begin{aligned}\frac{1}{2}\tau\big(\mathrm{sh}^2\,\frac{1}{2}\rho(x,y)\big) &\geq -c_n \log \mathrm{th}\,\frac{1}{4}\rho(x,y)\\ &= 2c_n\,\mathrm{arth}\left(e^{-\frac{1}{2}\rho(x,y)}\right) \geq 2c_n e^{-\frac{1}{2}\rho(x,y)}\,.\end{aligned}$$

Here the identities $2\,\mathrm{ch}^2 A = 1 + \mathrm{ch}\,2A$, $\mathrm{sh}\,2A = 2\,\mathrm{ch}\,A\,\mathrm{sh}\,A$, and $\log \mathrm{th}\, s = -2\,\mathrm{arth}\, e^{-2s}$ were applied. Recall that

$$\mathrm{sh}^2\,\frac{1}{2}\rho(x,y) = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}$$

by (3.12). Similarly, by 5.22(3) we obtain also

$$\begin{aligned}\frac{1}{2}\tau\big(\mathrm{sh}^2\,\frac{1}{2}\rho(x,y)\big) &\leq \frac{1}{2}c_n\,\mu\big(\mathrm{th}^2(\frac{1}{4}\rho(x,y))\big) < \frac{1}{2}c_n \log\frac{4}{\mathrm{th}^2\,\frac{1}{4}\rho(x,y)}\\ &= c_n \log\frac{2}{\mathrm{th}\,\frac{1}{4}\rho(x,y)}\,.\end{aligned}$$

6.10. Lemma. *Let G be a proper subdomain of $\mathbb{R}^n$, $x \in G$, $d(x) = d(x, \partial G)$, $B_x = B^n(x, d(x))$, let $y \in B_x$ with $y \neq x$, and let $r = |x-y|/d(x)$. Then the following two inequalities hold:*

$$(1)\qquad \lambda_G(x,y) \geq \lambda_{B_x}(x,y) = \frac{1}{2}\,\tau\Big(\frac{r^2}{1-r^2}\Big) > c_n \log\frac{1}{r}$$

$$(2)\qquad \mu_G(x,y) \leq \mu_{B_x}(x,y) = \gamma\Big(\frac{1}{r}\Big) \leq \omega_{n-1}\Big(\log\frac{1}{r}\Big)^{1-n}\,.$$

6.11. Lemma. *The inequality*

$$p(x) \geq \max\big\{\,\tau(|x|)\,,\,\tau(|x-e_1|)\big\}$$

holds for all $x \in \mathbb{R}^n \setminus \{0, e_1\}$. Equality holds if $x = se_1$ and $s < 0$ or $s > 1$.

The following theorem summarizes some properties of $p(x)$.

6.12. Theorem. *For $|x - e_1| \leq |x|$, $x \in \mathbb{R}^n \setminus \{0, e_1\}$*

$$\begin{aligned}&(1)\qquad p(x) \leq 2\,\tau(|x-e_1|) \text{ when } |x+e_1| \geq 2,\\ &(2)\qquad p(x) \leq 4\,\tau(|x-e_1|) \text{ when } |x| \geq 1,\\ &(3)\qquad p(x) \leq 2^{n+1}\tau(|x-e_1|).\end{aligned}$$

This result was improved by D. Betsakos [B] who proved the next theorem. The sharp constant in Theorem 6.13 is not known for $n > 2$, for $n = 2$, see [BV].

6.13. Theorem. *For $x \in \mathbb{R}^n \setminus \{0, e_1\}$*

$$p(x) \leq 4\tau(\min\{|x|, |x-e_1|\})\,. \tag{6.14}$$

For $x \in \mathbb{R}^n \backslash \{0\}$ we denote by r_x a similarity map with $r_x(0) = 0$ and $r_x(x) = e_1$. Then it is easy to see that $|r_x(y) - e_1| = |x-y|/|x|$. It follows immediately from the definitions (6.2) and (6.5) that

$$\lambda_{\mathbb{R}^n \backslash \{0\}}(x,y) = \min\{\, p(r_x(y))\,,\, p(r_y(x))\,\}\,. \tag{6.15}$$

Next we deduce the following two–sided inequality for $\lambda_{\mathbb{R}^n \backslash \{0\}}(x,y)$.

6.16. Theorem. *For distinct $x, y \in \mathbb{R}^n \setminus \{0\}$ the following inequality holds*

$$1 \,\le\, \lambda_{\mathbb{R}^n \backslash \{0\}}(x,y)/\tau\big(|x-y|/\min\{|x|,|y|\}\big) \,\le\, 4\,.$$

6.17. Corollary. *Let G be a proper subdomain of $\mathbb{R}^n$, x and y distinct points in G and $m(x,y) = \min\{d(x), d(y)\}$. Then*

$$\lambda_G(x,y) \le \inf_{z \in \partial G} \lambda_{\mathbb{R}^n \backslash \{z\}}(x,y) \le 4\,\tau\big(|x-y|/m(x,y)\big)\,.$$

Proof. The first inequality follows from 6.7. For the second one fix $z_0 \in \partial G$ with $m(x,y) = d(\{x,y\},\{z_0\})$. Applying 6.16 to $\mathbb{R}^n \setminus \{z_0\}$ yields the desired result. □

We next show that 6.17 fails to be sharp for a Jordan domain G in $\mathbb{R}^n$. For $t \in (0, \frac{1}{5})$ consider the family $G_t = B^n(-e_1, 1) \cup B^n(e_1, 1) \cup B^n(t)$ of Jordan domains. Then by 6.17

$$\lambda_{G_t}(-e_1, e_1) \le 4\,\tau(2)$$

for all $t \in (0, \frac{1}{5})$. But this is far from sharp because in fact

$$\begin{aligned} \lambda_{G_t}(-e_1, e_1) &\le \mathsf{M}\big(\Delta(\,[-2e_1, -e_1]\,,\,[e_1, 2e_1]\,;\,G_t)\big) \\ &\le \omega_{n-1}\Big(\log\frac{1}{t}\Big)^{1-n} \longrightarrow 0 \end{aligned}$$

as $t \to 0$. However, for a wide class of domains, which we shall now consider, the upper bound in 6.17 is essentially best possible.

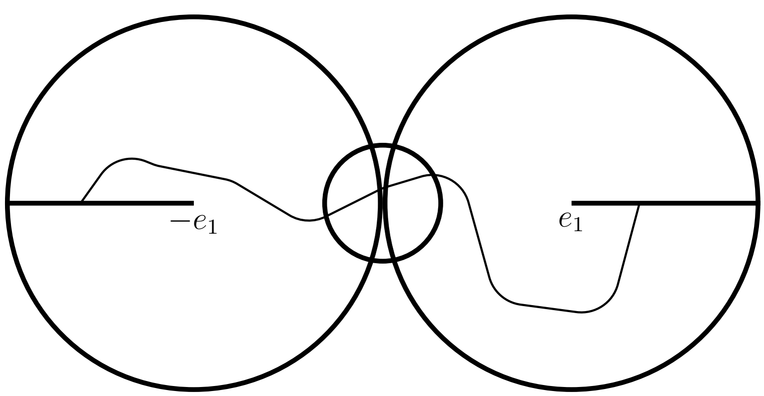

FIGURE 16. The family $\Delta(\,[-2e_1, -e_1]\,,\,[e_1, 2e_1]\,;\,G_t)\,.$

6.18. QED domains. A closed set E in $\overline{\mathbb{R}}^n$ is called a *c–quasiextremal distance set* or c-QED *exceptional set* or c-QED *set*, $c \in (0,1]$, if for each pair of disjoint continua $F_1,\ F_2 \subset \overline{\mathbb{R}}^n \setminus E$

$$\mathsf{M}\big(\Delta(F_1,F_2;\overline{\mathbb{R}}^n \setminus E)\big) \geq c\,\mathsf{M}\big(\Delta(F_1,F_2)\big)\ . \tag{6.19}$$

If G is a domain in $\overline{\mathbb{R}}^n$ such that $\overline{\mathbb{R}}^n \setminus G$ is a c-QED set, then we call G a c-QED domain.

6.20. Examples. (1) The unit ball $\mathbf{B}^n$ is a $\frac{1}{2}$–QED set by [GM1] or by the above Lemma 5.6.

(2) If E is a compact set of capacity zero, then E is a 1–QED set. For instance all isolated sets are 1–QED sets. The class of all 1–QED sets contains all closed sets in $\mathbb{R}^n$ of vanishing $(n-1)$–dimensional Hausdorff measure (see [V3], [GM1]).

(3) $\mathbf{B}^2 \setminus [0,e_1)$ is not a c-QED set for any $c>0$.

6.21. Theorem. *Let G be a c-QED domain in $\mathbb{R}^n$. Then*

$$\lambda_G(x,y) \geq c\,\tau(s^2+2s) \geq 2^{1-n}c\,\tau(s)$$

where $s = |x-y|/\min\{d(x),\, d(y)\}$.

It should be noted that the lower bound of 6.21 is very close to that of 6.16; in fact it differs only by a multiplicative constant.

In the next few theorems we shall give some estimates for the conformal metric μ_G.

6.22. Lemma. *Let G be a proper subdomain of $\mathbb{R}^n$, $s \in (0,1)$, $x,y \in G$. If $k_G(x,y) \leq 2\log(1+s)$, then*

(1) $\mu_G(x,y) \leq \gamma\Big(\dfrac{1}{\mathrm{th}(k_G(x,y)/(1-s))}\Big)$.

Moreover, there exist positive numbers b_1 and b_2 depending only on n such that

(2) $\mu_G(x,y) \leq b_1 k_G(x,y) + b_2$

for all $x,y \in G$.

It should be observed that (6.22(2)) is a generalization of the upper bound in (5.25) to the case of an arbitrary domain. The lower bound in (5.25) will next be generalized to the case of domains with connected boundary.

6.23. Lemma. *Let G be a domain in $\mathbb{R}^n$ such that ∂G is connected. Then for all $a,b \in G$, $a \neq b$,*

(1) $\mu_G(a,b) \geq \tau(4m^2+4m) \geq c_n\, j_G(a,b)$

where c_n is the constant in 5.8 *and $m = \min\{d(a),\, d(b)\}/|a-b|$. If, in addition, G is uniform, then*

(2) $\mu_G(a,b) \geq B\, k_G(a,b)$

for all $a,b \in G$.

7. Distortion theory

For the basic properties and definitions of K-quasiconformal and K-quasiregular mappings we refer the reader to [Vu3] as well as to the other papers in this same volume. See, in particular, Rasila's paper. One of the key ideas is that under a K-quasiconformal mapping, the modulus is changed at most by a constant $c \in [1/K, K]$. The notions introduced in the previous chapters enable us to formulate this basic property in a more concrete and geometric way, in terms of metrics.

Theorem 7.1 and Corollary 7.2 are the key results of this paper, and the other results in this section are just consequences. One should carefully observe that although the transformation rule in Corollary 7.2 looks like a bilipschitz property; the mappings need not be bilipschitz in the euclidean metric. This is because the metric μ_G behaves in a non-linear fashion. In the euclidean metric quasiconformal mappings are Hölder-continuous as the results below show.

7.1. Theorem. *If $f\colon G \to \mathbb{R}^n$ is a non–constant qr mapping, then*

(1) $\mu_{fG}(f(a), f(b)) \le K_I(f)\, \mu_G(a,b)\ ; \quad a, b \in G\ .$

In particular, $f\colon (G, \mu_G) \to (fG, \mu_{fG})$ is Lipschitz continuous. If $N(f,G) < \infty$, then

(2) $\lambda_G(a,b) \le K_O(f)\, N(f,G)\, \lambda_{fG}(f(a), f(b))$

for all $a, b \in G$ with $f(a) \ne f(b)$.

7.2. Corollary. *If $f\colon G \to G' = fG$ is a qc mapping, then*

$$(1) \quad \mu_G(a,b)\big/K_O(f) \le \quad \mu_{fG}(f(a), f(b)) \le K_I(f)\, \mu_G(a,b)\ ,$$

$$(2) \quad \lambda_G(a,b)\big/K_O(f) \le \quad \lambda_{fG}(f(a), f(b)) \le K_I(f)\, \lambda_G(a,b)$$

hold for all distinct $a, b \in G$.

7.3. Theorem. *Let $f\colon \mathbf{B}^n \to \mathbb{R}^n$ be a non–constant K–qr mapping with $f\mathbf{B}^n \subset \mathbf{B}^n$ and let $\alpha = K_I(f)^{1/(1-n)}$. Then*

$$(1) \quad \operatorname{th} \tfrac{1}{2}\rho\big(f(x), f(y)\big) \quad \le \varphi_K \left(\operatorname{th} \tfrac{1}{2}\rho(x,y)\right) \le \lambda_n^{1-\alpha} \left(\operatorname{th} \tfrac{1}{2}\rho(x,y)\right)^{\alpha}\ ,$$

$$(2) \quad \rho\big(f(x), f(y)\big) \quad \le K_I(f)\big(\rho(x,y) + \log 4\big)\ ,$$

hold for all $x, y \in \mathbf{B}^n$, where λ_n is the Grötzsch ring constant.

7.4. Corollary. *Let $f\colon \mathbf{B}^n \to \mathbf{B}^n$ be a K–qr mapping with $f(0) = 0$ and let $\alpha = K_I(f)^{1/(1-n)}$. Then*

$$(1) \quad |f(x)| \quad \le \varphi_{K,n}(|x|) \le \lambda_n^{1-\alpha} |x|^{\alpha} \le 2^{1-1/K} K |x|^{1/K}\ ,$$

$$(2) \quad |f(x)| \quad \le \frac{a-1}{a+1}\ , \quad a = \left(4\, \frac{1+|x|}{1-|x|}\right)^{K_I(f)}\ ,$$

for all $x \in \mathbf{B}^n$.

7.5. Example. Let $g\colon \mathbf{B}^2 \to \mathbf{B}^2 \setminus \{0\} = g\mathbf{B}^2$ be the exponential function $g(z) = \exp(\frac{z+1}{z-1})$, $z \in \mathbf{B}^2$. We shall show that $g\colon (\mathbf{B}^2, \rho) \to (g\mathbf{B}^2, k_{g\mathbf{B}^2})$ fails to be uniformly continuous. To this end, let $x_j = (e^j - 1)/(e^j + 1)$, $j = 1, 2, \ldots$. Then it follows that $\rho(0, x_j) = j$ and thus $\rho(x_j, x_{j+1}) = 1$. Let $Y = \mathbf{B}^2 \setminus \{0\}$. Since $g(x_j) = \exp(-e^j)$ we get by (4.4) and (3.20)

$$\begin{aligned} k_Y\big(g(x_j), g(x_{j+1})\big) &\geq j_Y\big(g(x_j), g(x_{j+1})\big) \\ &= \log\big[1 + \big(\exp e^{j+1}\big)\big(\exp(-e^j) - \exp(-e^{j+1})\big)\big] \\ &= \log\big[1 + \exp(e^{j+1} - e^j) - 1\big] = e^{j+1} - e^j \to \infty \end{aligned}$$

as $j \to \infty$. In conclusion, $g\colon (\mathbf{B}^2, \rho) \to (Y, k_Y)$ cannot be uniformly continuous, because $\rho(x_j, x_{j+1}) = 1$.

7.6. Theorem. *Let $f\colon \mathbf{B}^n \to \mathbb{R}^n$ be a non–constant qr mapping, let $E \subset \mathbb{R}^n \setminus f\mathbf{B}^n$ be a non–degenerate continuum such that $\infty \in E$, and let $G = \mathbb{R}^n \setminus E$ be a domain.*
(1) *Then $f\colon (\mathbf{B}^n, \rho) \to (G, j_G)$ is uniformly continuous.*
(2) *If G is uniform, then $f\colon (\mathbf{B}^n, \rho) \to (G, k_G)$ is uniformly continuous.*

7.7. Theorem. *Suppose that $f\colon G \to \mathbb{R}^n$ is a bounded qr mapping and that F is a compact subset of G. Let $\alpha = K_I(f)^{1/(1-n)}$ and $C = \lambda_n^{1-\alpha} d(fG)/d(F, \partial G)^\alpha$ where λ_n is the Grötzsch constant. Then f satisfies the Hölder condition*

$$|f(x) - f(y)| \leq C\,|x - y|^\alpha \tag{7.8}$$

for $x \in F$, $y \in G$.

7.9. Theorem. *Let $f\colon \mathbf{B}^n \to \mathbb{R}^n$ be a non–constant qr mapping.*
(1) *If $\varphi \in (0, \frac{1}{2}\pi)$ and $f\mathbf{B}^n \subset C(\varphi)$, then for all $x \in \mathbf{B}^n$*

$$|f(x)| \leq |f(0)|\, 4^{a\varphi} \Big(\frac{1 + |x|}{1 - |x|}\Big)^{a\varphi}$$

where a depends only on n and $K_I(f)$.
(2) *If $f\mathbf{B}^n \subset \{\, x \in \mathbb{R}^n : x_1^2 + \cdots + x_{n-1}^2 < 1 \,\}$, then for all $x, y \in \mathbf{B}^n$*

$$|f(x)| \leq |f(y)| + A\, K_I(f)\, (\rho(x, y) + \log 4)$$

where A is a positive constant depending only on n.

7.10. Theorem. *Let $f\colon \mathbf{B}^n \to \mathbf{B}^n$ be a qr mapping with $N(f, \mathbf{B}^n) = N < \infty$. Then*

$$\operatorname{th} \tfrac{1}{4}\rho\big(f(x), f(y)\big) \leq 2\,\big(\operatorname{th} \tfrac{1}{4}\rho(x, y)\big)^\beta$$

holds for all $x, y \in \mathbf{B}^n$ where $\beta = 1/(NK_O(f))$. Furthermore, if $f(0) = 0$, then for all $x \in \mathbf{B}^n$

$$\frac{|f(x)|}{1 + \sqrt{1 - |f(x)|^2}} \leq 2\Big(\frac{|x|}{1 + \sqrt{1 - |x|^2}}\Big)^\beta .$$

7.11. Exercise. Assume that $f\colon \mathbf{B}^n \to \mathbf{B}^n$ is K–qc with $f(0)=0$ and $f\mathbf{B}^n = \mathbf{B}^n$. Show that

$$\begin{aligned} |f(x)|^2 &\le \min\{\, \varphi^2_{K,n}(|x|)\,,\, 1-\varphi^2_{1/K,n}\big(\sqrt{1-|x|^2}\,\big)\,\}\,, \\ |f(x)|^2 &\ge \max\{\, \varphi^2_{1/K,n}(|x|)\,,\, 1-\varphi^2_{K,n}\big(\sqrt{1-|x|^2}\,\big)\,\}\,. \end{aligned}$$

Note that in the case $n=2$ we have $\varphi^2_{K,2}(r) = 1-\varphi^2_{1/K,2}(\sqrt{1-r^2}\,)$ for all $K>0$ and $0<r<1$, while the analogous relation fails to hold for $n\ge 3$.

The next result is a famous theorem of A. Mori from 1956 [LV]. The theorem has, however, one esthetic flaw: it is not sharp when $K=1$. It was conjectured in the 1960's that the constant 16 in the theorem could be replaced by $16^{1-1/K}$ and also shown in [LV] that this would be sharp. In 1988 it was proven in [FeV] that we can replace 16 by $M(K)\to 1$ as $K\to 1$. Perhaps the latest paper dealing with the problem of reducing the constant $M(K)$ was written by S.-L. Qiu [Q], but as far as we know it is still an open problem whether the constant $16^{1-1/K}$ could be achieved. Settling this problem would be remarkable progress, since a lot of work has been done. For the spherical chordal metric this problem was recently discussed by P. Hästö [H3].

7.12. Theorem. *Let $f\colon \mathbf{B}^2 \to \mathbf{B}^2$ be a K–qc mapping with $f(0)=0$ and $f\mathbf{B}^2=\mathbf{B}^2$. Then*

$$|f(x)-f(y)| \le 16\,|x-y|^{1/K}$$

for all $x,y\in\mathbf{B}^2$. Furthermore, the number 16 cannot be replaced by any smaller absolute constant.

7.13. An open problem. For $K\ge 1$, $n\ge 2$, and $r\in(0,1)$ let

$$\varphi^*_{K,n}(r) = \varphi^*_K(r) = \sup\{\, |f(x)| : f\in \mathcal{QC}_K(\mathbf{B}^n),\ f(0)=0,\ |x|\le r\,\}$$

where $\mathcal{QC}_K(\mathbf{B}^n) = \{\, f\colon \mathbf{B}^n\to f\mathbf{B}^n \mid f \text{ is } K\text{–qc and } f\mathbf{B}^n\subset\mathbf{B}^n\,\}$. As shown in [LV, p. 64]

$$\varphi^*_{K,2}(r) = \varphi_{K,2}(r) \le 4^{1-1/K}\, r^{1/K} \tag{7.14}$$

for each $r\in(0,1)$ and $K\ge 1$. By 7.4(1)

$$\varphi^*_{K,n}(r) \le \varphi_{K,n}(r) \le \lambda_n^{1-\alpha}\, r^{\alpha}\,,\quad \alpha = K^{1/(1-n)}\,, \tag{7.15}$$

for $n\ge 2$, $K\ge 1$, $r\in(0,1)$. A. V. Sychev [SY, p. 89] has conjectured that

$$\varphi^*_{K,n}(r) \le 4^{1-\alpha}\, r^{\alpha} \tag{7.16}$$

for all $n\ge 2$ and $K\ge 1$. Because $\lambda_2=4$, (7.16) agrees with (7.14) for $n=2$. In [AVV4] it is shown that $\varphi^*_{K,n}\not\equiv\varphi_{K,n}$ for $n\ge 3$. It follows from 7.10 and 7.11 that

$$\begin{cases} \varphi^*_{K,n}(r) \le 4\, r^{1/K}\,, \\ \big[\varphi^*_{K,n}(r)\big]^2 \le 1-\varphi^2_{1/K,n}(\sqrt{1-r^2}\,)\,. \end{cases} \tag{7.17}$$

From (7.17) and (7.15) it follows, as shown in [AVV], that

$$\varphi^*_{K,n}(r) \le 4^{1-1/K^2} r^{1/K} \tag{7.18}$$

holds for all $n \ge 2$, $K \ge 1$, $r \in (0,1)$. Note that the right hand side of (7.18) is bounded when $K \to \infty$. Recall that $\lambda_n \to \infty$ as $n \to \infty$ and that $\lambda_n^{1-\alpha} \le 2^{1-1/K} K$. Note that Sychev's conjecture (7.16) still remains open.

7.19. Another open problem. In [Vu4], the following problem was stated. Let $\mathcal{QC}_K(\mathbb{R}^n) = \{\, f\colon \mathbb{R}^n \to \mathbb{R}^n \mid f \text{ is } K\text{–qc and } f(e_1) = e_1 \,\}$. Is it true that

$$\sup\{|g(x)| : |x| = r, g \in \mathcal{QC}_K(\mathbb{R}^n)\} = \sup\{|f(-re_1)| : f \in \mathcal{QC}_K(\mathbb{R}^n)\}\,, \tag{7.20}$$

when $r > 0$? For $n = 2$ the answer is in the affirmative by [LVV].

8. Open problems

Assume that $G \subset \mathbb{R}^n$ is a proper subdomain. For what follows, we will be interested mainly in the cases when the domain is a member of some well-known class of domains. Some examples are uniform domains, QED-domains, domains with uniformly perfect (in the sense of Pommerenke [Su]) boundaries and quasiballs, i.e. domains G of the form $G = f\mathbf{B}^n$ for quasiconformal $f\colon \mathbb{R}^n \to \mathbb{R}^n$. We denote the class of domains with $\mathcal{D}$. Let us consider collection of metrics $\mathcal{M} = \{\alpha_G, h_G, j_G, k_G, \lambda_G^{1/(1-n)}, \mu_G, q, |\cdot|\}$ where h_G refers to the hyperbolic metric when $n = 2$. Interesting categories of mappings, we denote them by $\mathcal{C}$, are Hölder, Lipschitz, isometries, quasiisometries and identity mappings.

The problems that we list below are just examples. There are a great many variations, by letting the domain, mapping and metric independently vary over the categories $\mathcal{D}$, $\mathcal{C}$, and $\mathcal{M}$.

8.1. Convexity of balls and smoothness of spheres. Fix $m \in \mathcal{M}$. Does there exist constant $T_0 > 0$ such that $D_m(x,T) = \{z \in G\colon m(x,z) < T\}$, is convex (in euclidean geometry) for all $T \in (0, T_0)$? Is $\partial G_m(x,T)$ smooth for $T < T_0$?

For instance, in the case $m = k_G$ both of these problems seem to be open. In passing, we remark that it follows from (4.4) and Theorem 4.7 (2) that when the radius tends to 0, quasihyperbolic balls become more and more round. The quasihyperbolic metric is used as a tool for many applications, but very little about the metric itself is known. See the theses [MA] and [L] and also Lindén's paper in this volume.

8.2. Lipschitz-constant of identity mapping. For $x, y \in \mathbf{B}^n, x \ne y$, the following inequality holds [Vu3, (2.27)]

$$|x-y| \le 2\,\mathrm{th}\,\frac{\rho_{\mathbf{B}^n}(x,y)}{4} < \frac{\rho_{\mathbf{B}^n}(x,y)}{2}.$$

We may now regard this result as an inequality for the modulus of continuity of $id\colon (\mathbf{B}^n, \rho_{\mathbf{B}^n}) \to (\mathbf{B}^n, |\cdot|)$. Instead of considering the identity mapping we could now take any mapping in our category of mappings and consider the problem of estimating the modulus of continuity between any two metric spaces in our category of metric spaces, see [Vu1], [S]. We list several particular cases of our problem. For $G = \mathbb{R}^n \setminus \{0\}$ does there exist constants A or B such that for all $x, y \in G$

$$q(x,y) \le A k_G(x,y),$$

and

$$q(x,y) \le B \lambda_G^{1/(1-n)}(x,y)\,?$$

For $G = \mathbb{C} \setminus \{0,1\}$ does there exist constant C such that for all $x, y \in G$

$$q(x,y) \le C h_G(x,y)\,,$$

For $G = \mathbb{R}^n \setminus \{0\}$ does there exist a constant E such that for all $x, y \in G$

$$\lambda_G^{1/(1-n)}(x,y) \le E j_G(x,y)\,?$$

8.3. Characterization of isometries and quasiisometries. Given two metric spaces in our category of spaces, does there exist a quasiisometry, mapping the one space onto the other space? Again, we could consider, in place of quasiisometries, any other map in our category of maps.

What is the modulus of continuity of $id\colon (G, \mu_G) \to (G, \lambda_G^{1/(1-n)})$?

Is a quasiisometry $f\colon (G, \lambda_G^{1/(1-n)}) \to (fG, \lambda_{fG}^{1/(1-n)})$ quasiconformal? J. Lelong-Ferrand raised this question in [LF] and the question was answered in the negative in [FMV]. There it was also shown that the answer is affirmative under the stronger requirement that $f\colon (D, \lambda_D^{1/(1-n)}) \to (fD, \lambda_{fD}^{1/(1-n)})$ be uniformly continuous for all subdomains D of G. However, it is not known what the isometries are.

Are isometries $f\colon (G, \alpha_G) \to (fG, \alpha_{fG})$ Möbius transformations? (see Beardon [BE2], Hästö and Ibragimov [HI] and also Hästö's paper in this volume).

8.4. Conformal invariants. The conformal invariant $p(x)$ is relatively well-known. See [HV] for further information. However, much less is known about the invariants μ_G and λ_G. For domains whose boundaries are uniformly perfect (in the sense of Pommerenke), there are some inequalities for μ_G in terms of j_G, see [Vu2] and [JV]. Some results for λ_G when $G = \mathbf{B}^n \setminus \{0\}$, were proved in [H] and [BV]. But even the basic question of finding a formula for $\lambda_{\mathbf{B}^2\setminus\{0\}}(x,y)$ is open.

Some of these problems may be hard, some are very easy. Because of the very general setup, it would require some effort even to single out the interesting combinations of domains in $\mathcal{D}$, mappings in $\mathcal{C}$, and metrics in $\mathcal{M}$.

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