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Quasiminimizers and Potential Theory

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Abstract. Quasiminimizers are almost minimizers of variational integrals. Although quasiminimizers do not form a sheaf and do not provide a unique solution to the Dirichlet problem it is shown that they form an interesting basis for a potential theory. Quasisuperminimizers and their Poisson modifications are considered as well as their convergence properties. Special attention is devoted to the theory on the real line.

Keywords. quasiminimizers, quasisuperminimizers, quasisuperharmonic functions.

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Contents

1. Introduction

Quasiminimizers minimize a variational integral only up to a multiplicative constant. More precisely, let $\Omega \subset \mathbb{R}^n$ be an open set, $K \geq 1$ and $1 \leq p < \infty$. In the case of the p-Dirichlet integral, a function u belonging to the Sobolev space $W^{1,p}_{\mathrm{loc}}(\Omega)$ is a (p, K)-quasiminimizer or a K-quasiminimizer, if

$$\int_{\Omega'} |\nabla u|^p dx \leq K \int_{\Omega'} |\nabla v|^p \, dx \tag{1.1}$$

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for all functions $v \in W^{1,p}(\Omega')$ with $v - u \in W_0^{1,p}(\Omega')$ and for all open sets Ω' with a compact closure in Ω. A 1-quasiminimizer, called a minimizer, is a weak solution of the corresponding Euler equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0. \tag{1.2}$$

Clearly being a weak solution of (1.2) is a local property. However, being a K-quasiminimizer is not a local property as one-dimensional examples easily show. This indicates that the theory for quasiminimizers differs from the theory for minimizers and that there are some unexpected difficulties.

Quasiminimizers have been previously used as tools in studying the regularity of minimizers of variational integrals, see [GG1–2]. The advantage of this approach is that it covers a wide range of applications and that it is based only on the minimization of the variational integrals instead of the corresponding Euler equation. Hence regularity properties as Hölder continuity and L^p-estimates are consequences of the quasiminimizing property. It is an important fact that nonnegative quasiminimizers satisfy the Harnack inequality, see [DT].

Instead of using quasiminimizers as tools, the objective of these lectures is to show that quasiminimizers have a fascinating theory themselves. In particular, they form a basis for nonlinear potential theoretic model with interesting features. From the potential theoretic point of view quasiminimizers have several drawbacks: They do not provide unique solutions of the Dirichlet problem, they do not obey the comparison principle, they do not form a sheaf and they do not have a linear structure even when the corresponding Euler equation is linear. However, quasiminimizers form a wide and flexible class of functions in the calculus of variations under very general circumstances. Observe that the quasiminimizing condition (1.1) applies not only to one particular variational integral but the whole class of variational integrals at the same time. For example, if a variational kernel $F(x, \nabla u)$ satisfies

$$\alpha|h|^p \le F(x,h) \le \beta|h|^p \tag{1.3}$$

for some $0 < \alpha \le \beta < \infty$, then the minimizers of

$$\int F(x, \nabla u)\, dx$$

are quasiminimizers of the p-Dirichlet integral

$$\int |\nabla u|^p\, dx. \tag{1.4}$$

Hence the potential theory for quasiminimizers includes all minimizers of all variational integrals similar to (1.4). The essential feature of the theory is the control provided by the bounds in (1.3).

For example, the coordinate functions of a quasiconformal or, more generally, quasiregular mapping are quasiminimizers of the n-Dirichlet integral

$$\int |\nabla u|^n\, dx$$

in all dimensions $n = 2, 3, \dots$.

Recently quasiminimizers have been considered in metric measure spaces. This means that a metric space (X, d) is equipped with a Borel measure μ which satisfies some standard assumptions like the doubling property. The Sobolev space $W^{1,p}$ is replaced by the so called Newtonian space $N^{1,p}$ which for $\mathbb{R}^n$ and the Lebesgue measure reduces to $W^{1,p}$. We do not consider metric spaces here although most of the results hold in this case under appropriate conditions. For this theory see [KM2].

2. Case $n = 1$

For $n = 1$ the definition (1.1) can be written in the following form: Let (a, b) be an open interval in $\mathbb{R}$ and $u \in W^{1,p}_{\mathrm{loc}}(a, b)$. Then u is a (p, K)-quasiminimizer, or K-quasiminimizer for short, if for all closed intervals $[c, d] \subset (a, b)$

$$\int_c^d |u'|^p \, dx \le K \int_c^d |v'|^p \, dx \tag{2.1}$$

whenever $u - v \in W^{1,p}_0(c, d)$.

Now affine functions are minimizers, i.e. 1-quasiminimizers, for every $p \ge 1$. This fact can be easily deduced from the one dimensional version of (1.2) if $p > 1$. For $p = 1$ this is trivial. Moreover, affine functions are the only minimizers for $p > 1$. Thus choosing $v(x) = \alpha(x - c) + \beta$ where

$$\alpha = (u(d) - u(c))/(d - c), \quad \beta = u(c)$$

we see that $u - v \in W^{1,p}_0(c, d)$ and (2.1) yields

$$\int_c^d |u'|^p dx \le K \frac{|u(d) - u(c)|^p}{|d - c|^{p-1}}, \tag{2.2}$$

see [GG2]. The inequality (2.2) gives another definition for a K-quasiminimizer u: the function u is a locally absolutely continuous function in (a, b) that satisfies (2.2) on each subinterval $[c, d]$ of (a, b).

Observe that $u \in W^{1,p}(c, d)$ in a bounded open interval (c, d) means that u is absolutely continuous on $[c, d]$ with

$$\int_c^d |u'|^p dx < \infty. \tag{2.3}$$

If $u \in W^{1,p}(c, d)$ and $u - v \in W^{1,p}_0(c, d)$, then $v \in W^{1,p}(c, d)$ and $v(c) = u(c)$, $v(d) = u(d)$. Functions $u \in W^{1,p}_{\mathrm{loc}}(a, b)$ are simply locally absolutely continuous functions on (a, b) such that (2.3) holds in each subinterval $[c, d] \subset (a, b)$.

We leave the following lemma as an exercise.

Lemma 2.4. *Suppose that u is a (p,K)-quasiminimizer in (a,b). Then u is a monotone function. If $p>1$ then u is either strictly monotone or constant.*

The following lemma is more difficult to prove. It does not hold for $p=1$.

Lemma 2.5. *Let u be a (p,K)-quasiminimizer, $p>1$, in an interval (a,b). If $b<\infty$, then u has a continuous extension to b and (2.2) holds in all intervals $[c,d]\subset(a,b]$.*

Proof. We may assume that u is increasing, $b=1$, $[0,1]\subset(a,b]$ and $u(0)=0$.

Fix $0\le c<t<1$. Now

$$\begin{aligned}\int_c^t u'\,dx &\le (t-c)^{(p-1)/p}\left(\int_c^t u'^p\,dx\right)^{\frac{1}{p}} \le (t-c)^{(p-1)/p}\left(\int_0^t u'^p\,dx\right)^{\frac{1}{p}}\\ &\le K^{\frac{1}{p}}\frac{(t-c)^{\frac{p-1}{p}}}{t^{\frac{p-1}{p}}}\int_0^t u'\,dx = K^{\frac{1}{p}}\left(1-\frac{c}{t}\right)^{\frac{p-1}{p}}\int_0^t u'\,dx\end{aligned}$$

where we have used the Hölder inequality and (2.2). Next we choose $c=1-(2^pK)^{\frac{1}{1-p}}$. Then $0<c<1$ and letting $t\in(c,1)$ we obtain

$$\left(1-\frac{c}{t}\right)^{\frac{p-1}{p}}<(1-c)^{\frac{p-1}{p}}=\frac{1}{2K^{\frac{1}{p}}}.$$

The above inequalities yield

$$\int_0^t u'\,dx=\int_0^c u'\,dx+\int_c^t u'\,dx\le\int_0^c u'\,dx+\frac{1}{2}\int_0^t u'\,dx$$

and hence

$$u(t)=u(t)-u(0)=\int_0^t u'\,dx\le 2\int_0^c u'\,dx=2u(c).$$

Since u is increasing, letting $t\to1$ we obtain

$$u(b)=u(1)=\lim_{t\to1}u(t)\le 2u(c)<\infty$$

and the last assertion of the lemma now follows from (2.2).

Lemma 2.5 shows that the natural domain of definition for a 1-dimensional quasiminimizer is the closed interval $[a,b]$.

Example 2.6. The function $u(x)=x^\alpha$, $\alpha>1/2$, is a $(2,K)$-quasiminimizer in $[0,\infty)$ for $K=\alpha^2/(2\alpha-1)$. This is a rather easy computation. Note that $u(x)=x^{1/2}$ is not a $(2,K)$-quasiminimizer in $[0,\infty)$ (and in $(0,\infty)$) since u' does not belong to $L^2(0,1)$.

We will consider one dimensional quasiminimizers again in Chapter 5. The one dimensional case was first studied in [GG2].

3. Properties of quasiminimizers

We start with a basic regularity property.

Theorem 3.1. *Suppose that u is a (p, K)-quasiminimizer in $\Omega \subset \mathbb{R}^n$, $p > 1$. If $0 < r < R$ are such that the ball $B(x, 2R) \subset \Omega$, then*

$$\operatorname{osc}(u, B(x, r)) \leq C(r/R)^{\alpha} \operatorname{osc}(u, B(x, R))$$

where $C < \infty$ and $\alpha \in (0, 1]$ depend on p, n and K only.

In particular Theorem 3.1 implies that u is locally Hölder continuous in Ω.

Another important property is the Harnack inequality.

Theorem 3.2. *Let u be as in Theorem 3.1 and $u \geq 0$. Then in each ball $B(x, R)$ such that $B(x, 2R) \subset \Omega$*

$$\sup_{B(x,R)} u \leq C \inf_{B(x,R)} u$$

where the constant C depends on p, n and K only.

Since quasiminimizers are functions in $W^{1,p}_{\mathrm{loc}}(\Omega)$ only, Theorem 3.1 also means that they can be made continuous after redefinition on a set of measure zero.

For $p > n$, and hence for all $p > 1$ for $n = 1$, Theorem 3.1 follows from the Sobolev embedding lemma. Note that for $p = 1 = n$ quasiminimizers are continuous but they need not be Hölder continuous.

We do not prove Theorems 3.1 and 3.2 here. The proof for Theorem 3.2 in the case $n = 1$ is relatively simple, see [GG2]. For the proof of Theorem 3.1 the De Giorgi method can be used. The basic tool is the Sobolev type inequality

$$\left(\fint_{B(x,r)} |u|^t \, dx \right)^{1/t} \leq cr \left(\fint_{B(x,r)} |\nabla u|^p \, dx \right)^{1/p}$$

for functions $u \in W^{1,p}_0(B(x, r))$ where $t > p$. The main difficulty is to prove that u is locally essentially bounded. For the proof see [GG1], [GG2] and [KS]. In the paper [KS] metric measure spaces are considered and hence the regularity proof of [KS] uses minimal assumptions.

In the general case $n \geq 2$ the proof for Theorem 3.2 is rather complicated, see [DT] and [KS]. The proof makes use of the Krylov–Safonov covering argument [KSa]. Very recently it has turned out that the Moser method can be employed to prove Theorems 3.1 and 3.2 for quasiminimizers even in metric measure spaces, see [Ma].

In Potential Theory the Harnack inequality and Harnack's principle are essentially equivalent. From Theorem 3.1 and 3.2 it easily follows ($p > 1$): Suppose that (u_i) is an increasing sequence of K-quasiminimizers in a domain Ω. If $\lim u_i(x_0) < \infty$ at some point $x_0 \in \Omega$, then $\lim u_i$ is a K-quasiminimizer.

We will return to the proof of this fact in the next chapter and in Appendix 2.

4. Quasisuperminimizers, Poisson modifications and regularity

Let Ω be an open set in $\mathbb{R}^n$. A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is called a (p,K)-quasisuperminimizer, or a K-quasisuperminimizer, if

$$\int_{\Omega'} |\nabla u|^p \, dx \le K \int_{\Omega'} |\nabla v|^p \, dx \tag{4.1}$$

holds for all open $\Omega' \subset\subset \Omega$ and all $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ such that $v \ge u$ a.e. in Ω' and $v - u \in W^{1,p}_0(\Omega')$.

Remarks 4.2. (a) A 1-quasisuperminimizer is called a superminimizer.

(b) A superminimizer is a supersolution of the p-harmonic equation

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0,$$

i.e. u satisfies

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx \ge 0$$

for all non–negative $\varphi \in C^\infty_0(\Omega)$, see [HKM] for this theory. Observe that for $p = 2$ every superharmonic (in the classical sense) function u is a superminimizer provided that u belongs to $W^{1,2}_{\mathrm{loc}}(\Omega)$, however, a superharmonic function need not belong to $W^{1,2}_{\mathrm{loc}}(\Omega)$. For $n = 2$ the classical example is $u(x) = -\log |x|$ which is superharmonic in $\mathbb{R}^2$ but does not belong to $W^{1,2}(B(0,1))$. We return to this problem in Chapter 6.

(c) The inequality (4.1) can be replaced by several other inequalities, for example

$$\int_{\Omega' \setminus E} |\nabla u|^p \, dx \le K \int_{\Omega' \setminus E} |\nabla v|^p \, dx,$$

where $E \subset \Omega' \setminus \{u \ne v\}$ is any measurable set. For the list of these conditions see [B] and [KM2].

A function u is called a K-quasisubminimizer if $-u$ is a K-quasisuperminimizer.

Note that if u is a K-quasisuperminimimizer, then αu and $u + \beta$ are K-quasisuperminimizers when $\alpha \ge 0$ and $\beta \in \mathbb{R}$. However, the sum of two K-quasisuperminimizers need not be a K-quasisuperminimizer even in the case $p = 2$.

Lemma 4.3. *Suppose that u_i, $i = 1, 2$, are K_i-quasisuperminimizers in Ω. Then $\min(u_1, u_2)$ is a K-quasisuperminimizer in Ω with $K = \min(K_1K_2, K_1 + K_2)$.*

Proof. We prove that $u = \min(u_1, u_2)$ is a K-quasisuperminimizer with $K \le K_1K_2$; this inequality is important in applications. The proof for $K \le K_1 + K_2$ is similar, see [KM2]. To this end let $\Omega' \subset\subset \Omega$ be an open set and $v - u \in W^{1,p}_0(\Omega')$, $v = u$ in $\Omega \setminus \Omega'$ and $v \ge u$. Now

$$\int_{\Omega'} |\nabla u|^p\, dx = \int_{\{u_1\le u_2\}\cap\Omega'} |\nabla u_1|^p\, dx + \int_{\{u_1> u_2\}\cap\Omega'} |\nabla u_2|^p\, dx$$

and write $w = \max(\min(u_2, v), u_1)$. Then $w \ge u_1$ a.e. in Ω', $w - u_1 \in W_0^{1,p}(\Omega')$ and $w = u_1$ if $u_1 > u_2$. Thus the quasisuperharmonicity of u_1, see Remark 4.2 (c), yields

$$\begin{aligned}\int_{\{u_1\le u_2\}\cap\Omega'} |\nabla u_1|^p\, dx &\le K_1 \int_{\{u_1\le u_2\}\cap\Omega'} |\nabla w|^p\, dx \\ &= K_1 \int_{\{u_1\le u_2\}\cap\{v<u_2\}\cap\Omega'} |\nabla v|^p\, dx + K_1 \int_{\{u_1\le u_2\}\cap\{v\ge u_2\}\cap\Omega'} |\nabla u_2|^p\, dx.\end{aligned}$$

¿From these inequalities we obtain

$$\begin{aligned}\int_{\Omega'} |\nabla u|^p\, dx &\le K_1 \int_{\{u_1\le u_2\}\cap\{v<u_2\}\cap\Omega'} |\nabla v|^p\, dx \\ &\quad + K_1 \int_{\{u_1\le u_2\}\cap\{v\ge u_2\}\cap\Omega'} |\nabla u_2|^p\, dx + \int_{\{u_1> u_2\}\cap\Omega'} |\nabla u_2|^p\, dx \\ &\le K_1 \int_{\{u_1\le u_2\}\cap\{v<u_2\}\cap\Omega'} |\nabla v|^p\, dx + K_1 \int_{\{v\ge u_2\}\cap\Omega'} |\nabla u_2|^p\, dx.\end{aligned}$$

Note that $\Omega' \cap \{u_1 > u_2\} \subset \Omega' \cap \{v \ge u_2\}$. On the other hand $\max(u_2, v) - u_2 \in W_0^{1,p}(\Omega')$, $\max(u_2, v) \ge u_2$ and $\max(u_2, v) - u_2 = 0$ in $\{v \le u_2\}$ and hence the quasisuperharmonicity of u_2 implies

$$\int_{\{v\ge u_2\}\cap\Omega'} |\nabla u_2|^p\, dx \le K_2 \int_{\{v\ge u_2\}\cap\Omega'} |\nabla v|^p\, dx.$$

This together with the previous inequality completes the proof.

The following corollary is important.

Corollary 4.4. *Suppose that u is a K-quasisuperminimizer and h is a superminimizer in Ω. Then $\min(u, h)$ is a K-quasisuperminimizer in Ω.*

Remark 4.5. Lemma 4.3 and Corollary 4.4 are the counterparts of a property of classical superharmonic functions: If u_1 and u_2 are superharmonic, then $\min(u_1, u_2)$ is superharmonic.

Corollary 4.4 implies the necessity part of the following result. The other half follows from Theorem 4.14 below.

Lemma 4.6. *Suppose that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$. Then u is a K-quasisuperminimizer, $p > 1$, if and only if $\min(u, c)$ is a K-quasisuperminimizer for each $c \in \mathbb{R}$.*

The Poisson modification is an important tool in Potential Theory. In the classical case $p = 2$ this means the following: Let u be superharmonic in Ω and $u \in W^{1,2}_{loc}(\Omega)$ (this assumption is not actually needed). If $\Omega' \subset\subset \Omega$ is an open set let h be a minimizer (harmonic) in Ω' with boundary values u, i.e. $u - h \in W^{1,2}_0(\Omega')$. The Poisson modification of u in Ω' is defined as

$$P(u, \Omega') = \begin{cases} h & \text{in} \quad \Omega', \\ u & \text{in} \quad \Omega \setminus \Omega'. \end{cases} \tag{4.7}$$

Then $P(u, \Omega')$ is a superharmonic function in Ω, $P(u, \Omega') \le u$ and $P(u, \Omega')$ is harmonic in Ω'. This theory works well for superminimizers for all $p > 1$, see [HKM].

For quasisuperminimizers the above method does not work as above. In particular there exists a K-quasisuperminimizer (even a K-quasiminimizer) u such that the function $P(u, \Omega')$ in (4.7) is not a K'-quasisuperminimizer for any $K' < \infty$. The example is one dimensional and requires some computation. However, there are two replacements for $P(u, \Omega')$.

The first Poisson modification is a modification of (4.7). Let u be a K-quasisuperminimizer in Ω, $p > 1$, and $\Omega' \subset\subset \Omega$ an open set. Let h be the minimizer with boundary values u in Ω', i.e. $u - h \in W^{1,p}_0(\Omega')$. Observe that such a unique function h exists - this is a basic result in the theory of Sobolev spaces, see e.g. [HKM]. Let

$$P_1(u, \Omega') = \begin{cases} \min(u, h) & \text{in} \quad \Omega', \\ u & \text{in} \quad \Omega \setminus \Omega'. \end{cases} \tag{4.8}$$

Theorem 4.9. [KM2] *The function $P_1(u, \Omega')$ is K-quasisuperminimizer in Ω.*

By the construction of $P_1(u, \Omega')$, $P_1(u, \Omega') \le u$ in Ω. Note also that if u is a K-quasiminimizer, then $P_1(u, \Omega')$ is also a K-quasiminimizer in Ω' by Corollary 4.4.

Next we consider another possibility for a Poisson modification. For this we need to consider obstacle problems. The obstacle method is the most important method in the nonlinear potential theory. Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in W^{1,p}(\Omega)$. Write

$$\mathcal{K}^+_u(\Omega) = \{v \in W^{1,p}(\Omega) : v - u \in W^{1,p}_0(\Omega),\ v \ge u \ \text{ a.e. in } \ \Omega\}.$$

The class $\mathcal{K}^-_u$ is defined similarly but $v \ge u$ is replaced by $v \le u$. The following result is well-known.

Lemma 4.10. [HKM] *The obstacle problem*

$$\inf_{v \in \mathcal{K}^-_u(\Omega)} \int_\Omega |\nabla v|^p dx, \quad p > 1,$$

has a unique solution $u^- \in \mathcal{K}^-_u(\Omega)$. Moreover, u^- is a subminimizer and continuous if u is continuous.

A similar result holds for the class $\mathcal{K}_u^+(\Omega)$ and the solution is a superminimizer.

Suppose now that u is a K-quasisuperminimizer in Ω and $\Omega' \subset\subset \Omega$ is open. Let u^- be the solution to the $\mathcal{K}_u^-(\Omega')$-obstacle problem. Define

$$P_2(u,\Omega') = \begin{cases} u^- & \text{in} \quad \Omega', \\ u & \text{in} \quad \Omega \setminus \Omega'. \end{cases} \tag{4.11}$$

Theorem 4.12. *The function $P_2(u,\Omega')$ is a K-quasisuperminimizer in Ω, a subminimizer in Ω' and a K-quasiminimizer in Ω'.*

The proof for Theorem 4.12 is in Appendix 1.

Superharmonic functions in the classical potential theory are lower semicontinuous. It turns out that quasisuperminimizers can be defined pointwise and the resulting function is lower semicontinuous.

Theorem 4.13. [KM2] *Suppose that u is a K-quasisuperminimizer in Ω, $p > 1$. Then the function $u^* : \Omega \to (-\infty, \infty]$ defined by*

$$u^*(x) = \lim_{r\to 0} \operatorname*{ess\,inf}_{B(x,r)} u$$

is lower semicontinuous and $u^* = u$ a.e. (u and u^* are the same function in $W^{1,n}_{\text{loc}}(\Omega)$).

The proof for Theorem 4.13 is based on the De Giorgi method that is used to prove a weak Harnack inequality

$$\left(\fint_{B(x,r)} u^\sigma \, dx \right)^{1/\sigma} \le c \operatorname*{ess\,inf}_{B(x,3r)} u$$

for a K-quasisuperharmonic function $u \ge 0$. Here $B(x,5r) \subset \Omega$ and c and $\sigma > 0$ depend only on $n, p > 1$ and K.

The following is Harnack's principle for quasisuperminimizers.

Theorem 4.14. *Suppose that (u_i) is an increasing sequence of K-quasisuperminimizers in Ω, $p > 1$. If $u = \lim u_i$ is such that either*

(i) *u is locally bounded above or*
(ii) *$u \in W^{1,p}_{\text{loc}}(\Omega)$,*

then u is a K-quasisuperminimizer.

The proof for Theorem 4.14 is somewhat tedious. It is presented in Appendix 2.

5. More about $n = 1$

We take a closer look at the case $n = 1$. Recall that a superminimizer is a 1-quasisuperminimizer. The next result is easy to prove, see [MS].

Lemma 5.1. *Suppose that $u : [a, b] \to \mathbb{R}$ is a superminimizer, $p > 1$. Then u is a concave function.*

¿From Lemma 5.1 it follows that a superminimizer is a Lipschitz function in each interval $[c, d] \subset\subset (a, b)$. It need not be a Lipschitz function in $[a, b]$.

How regular are K-quasiminimizers and K-quasisuperminimizers? The answer is not known for $n \geq 2$ but for $n = 1$ some exact answers have been obtained.

Let $1 < p < \infty$ and let $\omega : [a, b] \to [0, \infty)$ be a weight in $L^1(a, b)$. Set

$$G_p(\omega) = \sup_I \left(\fint_I \omega^p \, dx \right)^{\frac{1}{p}} \left(\fint_I \omega \, dx \right)^{-1}$$

where the supremum is taken over all intervals $I \subset [a, b]$. If $G_p(\omega) < \infty$, then ω is said to belong to the G_p-class of Gehring.

For a non-constant quasiminimizer u in $[a, b]$ set

$$K_p(u) = \sup_{[c,d]} \int_c^d |u'|^p \, dx \frac{(d-c)^{p-1}}{|u(d) - u(c)|^p}$$

where the supremum is taken over all intervals $[c, d] \subset [a, b]$. In other words, $K_p(u)$ is the least constant in (2.2). The following lemma is immediate.

Lemma 5.2. *Let $u : [a, b] \to \mathbb{R}$ be absolutely continuous and non-constant with $u' \geq 0$ a.e. Then u is $K_p(u)$-quasiminimizer with exponent p, $p > 1$, if and only if u' belongs to the G_p-class with $G_p(u') = K_p(u)^{\frac{1}{p}}$.*

A. Korenovskii [K] has determined the optimal higher integrability bound $p_0 = p_0(p, K)$ for a weight ω in the G_p-class. Hence Lemma 5.2 enables us to determinate the optimal integrability bound for the derivative of a K_p-quasiminimizer. Let $\gamma_{p,t} : [p, \infty) \to \mathbb{R}$, $p > 1$, $t > 1$, be the function

$$\gamma_{p,t}(x) = 1 - t^p \frac{x-p}{x} \left(\frac{x}{x-1}\right)^p,$$

and let $p_1(p, t) \in (p, \infty)$ be the unique solution of the equation $\gamma_{p,t}(x) = 0$. For the properties of $p_1(p, t)$ see [DS, Section 3].

Theorem 5.3. *Suppose that u is a (p, K)-quasiminimizer, $p \geq 1$, $K \geq 1$, in $[a, b]$. Then $u' \in L^s(a, b)$ for $1 \leq s < p_1(p, K^{\frac{1}{p}})$ if $p > 1$ and $K > 1$, $u' \in L^1(a, b)$ if $p = 1$ and $u' \in L^\infty(a, b)$ if $p > 1$ and $K = 1$. All these integrability conditions are sharp.*

Proof. Let first $p > 1$ and $K > 1$. We may assume that u is increasing. By Lemma 5.2, $G_p(u') = K_p(u)^{\frac{1}{p}}$ and from [K, Theorem 2] we conclude that $u' \in L^s(a,b)$ for $1 \le s < p_1(p, G_p(u')) = p_1(p, K_p(u)^{\frac{1}{p}})$.

If $p > 1$ and $K = 1$, then u is affine and hence $u' \in L^\infty(a,b)$. For $p = 1$, u' trivially belongs to $L^1(a,b)$.

To see that the bound $\alpha = p_1(p, K^{\frac{1}{p}})$ is sharp for $p > 1$ and $K > 1$ it suffices to consider the interval $[0,1]$. The function

$$u(x) = \frac{\alpha}{\alpha - 1} x^{\frac{\alpha-1}{\alpha}}, \quad x \in [0,1],$$

has the derivative $u'(x) = x^{-\frac{1}{\alpha}}$ and a direct computation shows that u' belongs to the Gehring class with $G_p(u') = K^{\frac{1}{p}}$, see [DS, Proposition 2.3]. By Lemma 5.2 u is a K-quasiminimizer. On the other hand, u' does not belong to $L^\alpha(0,1)$. This shows that the open ended upper bound $p_1(p, K^{\frac{1}{p}})$ is sharp.

For $p = 1$ the integrability of u' cannot be improved since every increasing absolutely continuous function u is a 1-quasiminimizer. The theorem follows.

Remark 5.4. For $p = 2$,

$$p_1(2,t) = 1 + t(t^2 - 1)^{-\frac{1}{2}}, \quad t > 1,$$

and hence

$$p_1(2, K^{\frac{1}{2}}) = 1 + K^{\frac{1}{2}}(K-1)^{-\frac{1}{2}}, \quad K > 1.$$

In [MS] the inverse functions of one dimensional quasiminimizers are also considered.

6. Quasisuperharmonic functions

In the nonlinear potential theory superharmonic functions can be defined in many ways. The most natural definition uses the comparison principle, see (6.3) below. Let $p > 1$ and let Ω be an open set in $\mathbb{R}^n$. A function $u : \Omega \to (-\infty, \infty]$ is said to be superharmonic, i.e. $(p,1)$-superharmonic, if

(6.1) u is lower semicontinuous,
(6.2) $u \not\equiv \infty$ in any component of Ω,
(6.3) for each open set $\Omega' \subset\subset \Omega$ and each minimizer $h \in C(\overline{\Omega}')$, i.e. $(p,1)$-quasiminimizer, the inequality $u \ge h$ on $\partial\Omega'$ implies $u \ge h$ in Ω'.

For the theory of superharmonic functions in the nonlinear situation see [HKM].

Superharmonic functions can also be defined with the help of minimizers, see [HKM, Theorem 7.10] and [HKM, Corollary 7.20]. For other definitions see [B].

Lemma 6.4. *Suppose that $u : \Omega \to (-\infty, \infty]$ satisfies* (6.1) *and* (6.2). *Then u is $(p,1)$-superharmonic if and only if there is an increasing sequence (u_i^*) of $(p,1)$-quasisuperminimizers, i.e. superminimizers, with $u = \lim u_i^*$. Here u_i^* is the lower semicontinuous representative of a superminimizer u_i.*

Note that given a superharmonic function u the sequence $u_i^* = \min(u, i)$, $i = 1, 2, \ldots$, is the required sequence.

In view of Lemma 6.4 the following definition for (p, K)-quasisuperharmonicity is natural.

Definition 6.5. Let $\Omega \subset \mathbb{R}^n$ be an open set, $p > 1$ and $K \geq 1$. A function $u : \Omega \to (-\infty, \infty]$ is said to be (p, K)-quasisuperharmonic if there is an increasing sequence of K-quasisuperminimizers u_i in Ω such that $\lim u_i^* = u$ and $u \not\equiv \infty$ in each component of Ω.

Lemma 6.6. *Suppose that u is a (p, K)-quasisuperharmonic function in Ω and locally bounded above. Then u is a (p, K)-quasisuperminimizer.*

Proof. By the definition for quasisuperharmonicity there is an increasing sequence of quasisuperminimizers $u_i^* : \Omega \to (-\infty, \infty)$ such that $\lim u_i^* = u$. From Theorem 4.14 it follows that u is a K-quasisuperminimizer as required.

Note that a (p, K)-quasisuperharmonic function is automatically lower semicontinuous as a limit of an increasing sequence of lower semicontinuous functions.

Lemma 6.7. *Let u be a K-quasisuperharmonic function in Ω and h a (continuous) minimizer in Ω. Then* $\min(u, h)$ *is a K-quasisuperminimizer (and hence K-quasisuperharmonic) in Ω.*

Proof. Let u_i^* be an increasing sequence of K-quasisuperminimizers in Ω such that $u_i^* \to u$. Now $\min(u_i^*, h)$ is lower semicontinuous and by Corollary 4.4, $\min(u_i^*, h)$ is a K-quasisuperminimizer. Since $\min(u_i^*, h) \leq h$, it follows that $\min(u, h) = \lim \min(u_i^*, h)$ is a K-quasisuperharmonic function. By Lemma 6.6, $\min(u, h)$ is a K-quasisuperminimizer.

Theorem 6.8. *Suppose that $u : \Omega \to (-\infty, \infty]$ satisfies* (6.1) *and* (6.2). *Then u is a K-quasisuperharmonic if and only if* $\min(u, c)$ *is a K-quasisuperminimizer for each $c \in \mathbb{R}$.*

Proof. The only if part follows from Lemma 6.7. For the sufficiency choose $c = i$, $i = 1, 2, \ldots$. Then $\min(u, i)$ is a lower semicontinuous K-quasisuperminimizer and it follows from Definition 6.5 that u is a K-quasisuperharmonic function.

The theory for K-quasisuperharmonic functions is still in its infancy. However, the following result was proved in [KM2]: A set $C \subset \mathbb{R}^n$ is said to be (p, K)-polar, if there is a neighborhood Ω of C and a (p, K)-quasisuperharmonic function u in Ω such that $u(x) = \infty$ for each $x \in C$. Then C is a (p, K)-polar set if and only if the p-capacity of C is zero. It has been previously known that a set $C \subset \mathbb{R}^n$ is a $(p, 1)$-polar set if and only if the p-capacity of C is zero. Hence allowing the freedom due to $K \geq 1$ adds nothing new to the structure of polar sets.

7. Appendix 1

Proof for Theorem 4.12. That $P_2(u, \Omega')$ is a subminimizer in Ω' follows from Lemma 4.9. In order to show that $w = P_2(u, \Omega')$ is a K-quasisuperminimizer in Ω let $\Omega'' \subset\subset \Omega$ be open and v a function such that $v - w \in W_0^{1,p}(\Omega'')$ and $v \geq w$ in Ω''. We set $v = w$ in $\Omega \setminus \Omega''$. For the K-quasisuperminimizing property of w it suffices to show

$$\text{(a)} \qquad \int_{\Omega''\cap\{w<v\}} |\nabla w|^p dx \leq K \int_{\Omega''\cap\{w<v\}} |\nabla v|^p dx.$$

Hence we may assume that $w < v$ in Ω'' although $\Omega'' \cap \{w < v\}$ need not be an open set. Write $A = \{x \in \Omega : u(x) < v(x)\}$. Then $A \subset \Omega''$ because if $x \in A \setminus \Omega''$, then $u(x) < v(x) = w(x)$ which is a contradiction since $u \geq w$. The quasisuperminimizing property of u yields

$$\text{(b)} \qquad \int_A |\nabla u|^p dx \leq K \int_A |\nabla v|^p dx,$$

see Remark 4.2 (c). The function $\min(u, v)$ satisfies $w \leq \min(u, v) \leq u$ in Ω and $\min(u, v)$ can be continued as w to $\Omega'' \setminus \{w < u\}$. The resulting function is in the right Sobolev space. Note also that $\min(u, v) = w$ outside $\Omega'' \cap \Omega'$ and that $\min(u, v)$ and w coincide outside $\{w < u\} \cap \Omega''$ in Ω. Since w is the solution to the $\mathcal{K}_u^-(\Omega')$-obstacle problem, $w \leq \min(u, v)$ and $\min(u, v)$ has the correct boundary values w in $\{w < u\} \cap \Omega''$, we obtain

$$\text{(c)} \qquad \begin{aligned} \int_{\{w<u\}\cap\Omega''} |\nabla w|^p \, dx &\leq \int_{\{w<u\}\cap\Omega''} |\nabla \min(u, v)|^p \, dx \\ &= \int_{\{w<u\}\cap\Omega''\cap\{u<v\}} |\nabla u|^p \, dx + \int_{\{w<u\}\cap\Omega''\cap\{u\geq v\}} |\nabla v|^p \, dx. \end{aligned}$$

Since

$$(\Omega'' \cap \{w = u\}) \cup (\{w < u\} \cap \Omega'' \cap \{u < v\}) \subset \Omega'' \cap \{u < v\},$$

the inequalities (b) and (c) yield

$$\begin{aligned} \int_{\Omega''} |\nabla w|^p dx &= \int_{\Omega''\cap\{w=u\}} |\nabla u|^p \, dx + \int_{\Omega''\cap\{w<u\}} |\nabla w|^p \, dx \\ &\leq \int_{\Omega''\cap\{w=u\}} |\nabla u|^p \, dx + \int_{\{w<u\}\cap\Omega''\cap\{u<v\}} |\nabla u|^p \, dx \\ &\quad + \int_{\{w<u\}\cap\Omega''\cap\{u\geq v\}} |\nabla v|^p \, dx \\ &\leq \int_{\Omega''\cap\{u<v\}} |\nabla u|^p \, dx + \int_{\{w<u\}\cap\Omega''\cap\{u\geq v\}} |\nabla v|^p \, dx \\ &\leq K \int_{\Omega''\cap\{u<v\}} |\nabla v|^p \, dx + \int_{\{w<u\}\cap\Omega''\cap\{u\geq v\}} |\nabla v|^p \, dx \leq K \int_{\Omega''} |\nabla v|^p \, dx. \end{aligned}$$

This is (a). We leave to an exercise to show that w is a K-quasiminimizer in Ω'. The proof is complete.

8. Appendix 2

Proof for Theorem 4.14. We show that the quasisuperminimizing property is preserved under the increasing convergence if the limit is locally bounded above or belongs to $W^{1,p}_{\text{loc}}(\Omega)$.

The proof of this theorem [KM2, Theorem 6.1] contains a gap which will be settled here. The argument is quite similar as in [KM2]. The authors would like to thank professor Fumi–Yuki Maeda for pointing out the error in the original paper.

We consider the case (i) only. The case (ii) follows from (i) and from an easy truncation argument, see [KM2, p. 477]. In the case (i) it follows from the De Giorgi type upper bound

$$\int\limits_{B(x,\rho)} |\nabla u_i|^p\,dx \le c(R-\rho)^{-p} \int\limits_{B(x,R)} (u_i - k)^p\,dx,$$

where

$$k < -\sup\{\operatorname*{ess\,sup}_{B(x,R)} u_i :\ i = 1, 2, \ldots\},$$

$0 < \rho < R$ and $B(x,R) \subset\subset \Omega$, that the sequence $(|\nabla u_i|)$ is uniformly bounded in $L^p(\Omega')$ for every $\Omega' \subset\subset \Omega$. This implies that $u \in W^{1,p}_{\text{loc}}(\Omega)$ and we may assume that $(|\nabla u_i|)$ converges weakly to ∇u in $L^p(\Omega')$.

Let $C \subset \Omega$ be a compact set and for $t > 0$ write

$$C(t) = \{x \in \Omega : \operatorname{dist}(x, C) < t\}.$$

Then $C(t) \subset\subset \Omega$ for $0 < t < \operatorname{dist}(C, \partial\Omega) = t_0$.

Lemma 8.1. *Let u and u_i be as in Theorem 4.12. Then for almost every $t \in (0, t_0)$ we have*

$$\limsup_{i\to\infty} \int\limits_{C(t)} |\nabla u_i|^p\,dx \le c \int\limits_{C(t)} |\nabla u|^p\,dx$$

where the constant c depends only on K and p.

Proof. Let $0 < t' < t < t_0$ and choose a Lipschitz cut-off function η such that $0 \le \eta \le 1$, $\eta = 0$ in $\Omega \setminus C(t)$ and $\eta = 1$ in $C(t')$. Let

$$w_i = u_i + \eta(u - u_i), \quad i = 1, 2, \ldots .$$

Then $w_i - u_i \in W_0^{1,p}(C(t))$ and $w_i \geq u_i$ a.e. in $C(t)$. Hence the quasisuperminimizing property of u_i gives

$$\begin{aligned}\int_{C(t')} |\nabla u_i|^p\,dx &\leq \int_{C(t)} |\nabla u_i|^p\,dx \leq K\int_{C(t)} |\nabla w_i|^p\,dx \\ &\leq \alpha K\Bigg(\int_{C(t)} (1-\eta)^p|\nabla u_i|^p\,dx \\ &\qquad + \int_{C(t)} |\nabla\eta|^p(u-u_i)^p\,dx + \int_{C(t)} \eta^p|\nabla u|^p\,dx\Bigg),\end{aligned}$$

where $\alpha = 2^{p-1}$. Adding the term

$$\alpha K \int_{C(t')} |\nabla u_i|^p\,dx$$

to the both sides and taking into account that $\eta = 1$ in $C(t')$ we obtain

$$\begin{aligned}(1+\alpha K)\int_{C(t')} |\nabla u_i|^p\,dx &\leq \alpha K\int_{C(t)} |\nabla u_i|^p\,dx \\ &\qquad +\alpha K\int_{C(t)} |\nabla\eta|^p(u-u_i)^p\,dx + \alpha K\int_{C(t)} |\nabla u|^p\,dx.\end{aligned}$$

Set $\Psi : (0,t_0) \to \mathbb{R}$,

$$\Psi(t) = \limsup_{i\to\infty} \int_{C(t)} |\nabla u_i|^p\,dx.$$

Now $-u_i$ belongs to the De Giorgi class (see [KM2, Lemma 5.1]), and hence Ψ is a finite valued and increasing function of t. Hence the points of discontinuity form a countable set. Let t, $0 < t < t_0$, be a point of continuity of Ψ. Letting $i \to \infty$, we obtain from the previous inequality the estimate

$$(1+\alpha K)\Psi(t') \leq \alpha K\Psi(t) + \alpha K\int_{C(t)} |\nabla u|^p\,dx,$$

because

$$\int_{C(t)} |\nabla\eta|^p(u-u_i)^p\,dx \to 0$$

as $i \to \infty$ by the Lebesgue monotone convergence theorem. Since t is a point of continuity of Ψ, we conclude that

$$(1+\alpha K)\Psi(t) \leq \alpha K\Psi(t) + \alpha K\int_{C(t)} |\nabla u|^p\,dx,$$

or in other words

$$\Psi(t) \le \alpha K \int_{C(t)} |\nabla u|^p \, dx.$$

Proof of Theorem 4.14, case (i). As noted before $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$. Let $\Omega' \subset\subset \Omega$ be open and $v \in W^{1,p}(\Omega')$, $v \ge u$ almost everywhere and $v - u \in W^{1,p}_0(\Omega')$. By [KM2, Lemma 6.2] it suffices to show that

$$(a) \qquad \int_{\Omega'} |\nabla u|^p \, dx \le \int_{\Omega'} |\nabla v|^p \, dx.$$

To this end let $\varepsilon > 0$ and choose open sets Ω'' and Ω_0 such that

$$\Omega' \subset\subset \Omega'' \subset\subset \Omega_0 \subset\subset \Omega$$

and

$$(b) \qquad \int_{\Omega_0 \setminus \overline{\Omega'}} |\nabla u|^p \, dx < \varepsilon.$$

Next choose a Lipschitz cut-off function η with the properties $\eta = 1$ in a neighborhood of $\overline{\Omega'}$, $0 \le \eta \le 1$ and $\eta = 0$ on $\Omega \setminus \Omega''$. Set

$$w_i = u_i + \eta(v - u_i), \quad i = 1, 2, \dots .$$

Then $w_i - u_i \in W^{1,p}_0(\Omega'')$ and $w_i \ge u_i$. Thus

$$\begin{aligned}
\Big(\int_{\Omega''} |\nabla w_i|^p \, dx\Big)^{1/p} &\le \Big(\int_{\Omega''} ((1-\eta)|\nabla u_i|^p + \eta|\nabla v|)^p \, dx\Big)^{1/p} \\
&\quad + \Big(\int_{\Omega''} |\nabla \eta|^p (v - u_i)^p \, dx\Big)^{1/p} \\
&= \alpha_i + \beta_i.
\end{aligned}$$

The elementary inequality

$$(\alpha_i + \beta_i)^p \le \alpha_i^p + p\beta_i(\alpha_i + \beta_i)^{p-1}$$

implies that

$$(c) \quad \int_{\Omega''} |\nabla w_i|^p \, dx \le \int_{\Omega''} (1-\eta)|\nabla u_i|^p \, dx + \int_{\Omega''} \eta |\nabla v|^p \, dx + p\beta_i(\alpha_i + \beta_i)^{p-1},$$

where we also used the convexity of the function $t \mapsto t^p$. We estimate the terms on the right-hand side separately.

Since $\eta = 1$ in a neighborhood of $\overline{\Omega'}$, there is a compact set $C \subset \overline{\Omega''}$ such that $C \cap \overline{\Omega'} = \emptyset$ and

$$\int_{\Omega''} (1-\eta)|\nabla u_i|^p \, dx \le \int_C |\nabla u_i|^p \, dx.$$

We can choose $C = \overline{\Omega''} \setminus \Omega'(t)$ for sufficiently small $t > 0$. Next choose $t > 0$ such that

$$\limsup_{i\to\infty} \int_{C(t)} |\nabla u_i|^p \, dx \le c \int_{C(t)} |\nabla u|^p \, dx$$

and $C(t) \subset \Omega_0 \setminus \overline{\Omega'}$. This is possible by Lemma 8.1. We have

$$\begin{aligned}\limsup_{t\to\infty} \int_{\Omega''} (1-\eta)|\nabla u_i|^p \, dx &\le \limsup_{i\to\infty} \int_{C} |\nabla u_i|^p \, dx \\ &\le \limsup_{i\to\infty} \int_{C(t)} |\nabla u_i|^p \, dx \le c \int_{C(t)} |\nabla u|^p \, dx \le c\varepsilon\end{aligned}$$

where the last inequality follows from (b). Since $\nabla\eta = 0$ in Ω' and $v = u$ in $\Omega'' \setminus \Omega'$ the Lebesgue monotone convergence theorem yields $\beta_i \to 0$ as $i \to \infty$. On the other hand the numbers α_i remain bounded as $i \to \infty$. Hence it follows from (c) that

$$\limsup_{i\to\infty} \int_{\Omega''} |\nabla w_i|^p \, dx \le c\varepsilon + \int_{\Omega''} \eta |\nabla v|^p \, dx \le c\varepsilon + \int_{\Omega''} |\nabla v|^p \, dx.$$

Now u_i is a K-quasisuperminimizer and hence for large i it follows that

$$\begin{aligned}\int_{\overline{\Omega'}} |\nabla u_i|^p \, dx &\le \int_{\Omega''} |\nabla u_i|^p \, dx \le K \int_{\Omega''} |\nabla w_i|^p \, dx \le 2Kc\varepsilon + K \int_{\Omega''} |\nabla v|^p \, dx \\ &\le 2Kc\varepsilon + K \int_{\overline{\Omega'}} |\nabla v|^p \, dx + K \int_{\Omega''\setminus\overline{\Omega'}} |\nabla v|^p \, dx \\ &\le 3Kc\varepsilon + K \int_{\overline{\Omega'}} |\nabla v|^p \, dx,\end{aligned}$$

where we used (b) and the fact that $\nabla u = \nabla v$ in $\Omega'' \setminus \overline{\Omega'}$. Since $\varepsilon > 0$ was arbitrary and since

$$\int_{\overline{\Omega'}} |\nabla u|^p \, dx \le \liminf_{i\to\infty} \int_{\overline{\Omega'}} |\nabla u_i|^p \, dx$$

by the weak convergence $\nabla u_i \to \nabla u$ in $L^p(\overline{\Omega'})$, this completes the proof of (a) and the proof for the case (i) is complete.

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