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Isometries of relative metrics

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Abstract. This survey contains a review of the basics of Möbius mappings and some basic metrics used in relations with quasiconformal mappings, the j_G metric and the quasihyperbolic metric. The second and third sections are devoted to the study of the isometries of these two metrics.

Keywords. Relative metrics, isometries, distance ratio metric, Seittenranta's metric.

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Contents

1. Overview

These lecture notes consist of three parts: In the first part the basic theory of Möbius mappings is reviewed. Particular emphasis will be given to concrete calculations within the context of a single mapping in Euclidean space. Although this presentation is perhaps not the most elegant one possible, it has the advantage that it does a good job in preparing us for the isometry questions that come up later. For a more detailed exposition of the basics of Möbius mappings see e.g. [2, 31].

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The latter two parts deal with the problem of characterizing isometries of two metrics which have turned out to be very important in the theory of quasiconformal mappings, namely, the j_G and the quasihyperbolic metric. Specifically, in the second part we deal with the j_G metric — this part is based on joint work with Z. Ibragimov and H. Lindén [18] in Computational Methods and Function Theory. The third part reproduces parts of my recent manuscript [15], which deals with isometries of the quasihyperbolic metric.

Characterizing isometries of a metric can in some sense be thought of as solving a (system of) functional equation(s): we know that

$$d_{f(G)}(f(x), f(y)) = d_G(x, y)$$

for all $x, y \in G$ and we want to determine f. However, the fact that we have at our disposal a continuum of functional equations implies that the methods used to approach this problem are somewhat different than those usually found when dealing with functional equations. Thus our methods will often be based on some geometric considerations: we will employ geodesics (locally and globally), intrinsic curvature, as well as limiting behavior of the metric in infinitesimal regions more generally.

Many other properties of these and related metrics have also been studied. A review of some of these results is presented in the chapter by H. Lindén in these notes.

2. Möbius mappings

We denote by $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ the one-point compactification of $\mathbb{R}^n$, so its open balls are the open balls of $\mathbb{R}^n$, complements of closed balls in $\mathbb{R}^n$ and half-spaces. If $D \subset \overline{\mathbb{R}^n}$ we denote by ∂D and $\overline{D}$ its boundary and closure, respectively, all with respect to $\overline{\mathbb{R}^n}$. By $B^n(x, r)$ and $S^{n-1}(x, r)$ we denote the open ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$, and its boundary, respectively. For $x \in D \subsetneq \mathbb{R}^n$ we denote $\delta(x) = d(x, \partial D) = \min\{|x - z| \colon z \in \partial D\}$. By $[x, y]$, $(x, y]$ we denote the closed and half-open segment between x and y, respectively.

The (absolute) cross-ratio of four distinct points is defined by

$$|a, b, c, d| = \frac{|a - c|\,|b - d|}{|a - b|\,|c - d|},$$

with the understanding that $\frac{|\infty - x|}{|\infty - y|} = 1$ for all $x, y \in \mathbb{R}^n$. A homeomorphism $f \colon \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ is a Möbius mapping if

$$|f(a), f(b), f(c), f(d)| = |a, b, c, d|$$

for every quadruple of distinct points $a, b, c, d \in \overline{\mathbb{R}^n}$. A mapping of a subdomain of $\overline{\mathbb{R}^n}$ is Möbius, if it is a restriction of a Möbius mapping defined on $\overline{\mathbb{R}^n}$.

Although the previous definition is very compact and brings out one important aspect of Möbius mappings, it does not tell us what the behavior of a Möbius mapping is in terms of geometry. However, it is not so difficult to get some results in this direction: Let us regard three points a, b, c as fixed and a fourth point x as variable. Then the cross-ratio equation reads

$$\frac{|a-c|}{|a-b|}\frac{|b-x|}{|c-x|} = \frac{|a'-c'|}{|a'-b'|}\frac{|b'-x'|}{|c'-x'|},$$

where a' is the image of a under the Möbius mapping. We can rewrite this as

$$\frac{|b-x|}{|c-x|} = C\frac{|b'-x'|}{|c'-x'|},$$

where C is a constant not depending on x. However, for fixed b, c and $C > 0$ the set

$$\Big\{x \in \mathbb{R}^n \colon \frac{|b-x|}{|c-x|} = C\Big\}$$

is a sphere. Thus the previous equation implies that the Möbius mapping maps spheres to spheres. The converse of this statement is also true, see [4].

It is also possible to take a more constructive approach to Möbius mappings. Let us first of all make the trivial observation that a mapping which preserves Euclidean distances is Möbius. Second, we note that mappings preserving ratios of Euclidean distances (so-called similarity mappings) are Möbius. These mappings are:

- translations;
- reflections;
- rotations; and
- dilatations.

Are there any other Möbius mappings?

From the definition it is clear that the set of Möbius mappings is closed under composition (in fact, the set is a group under composition). Thus we may employ a very useful trick in trying to identify any other Möbius mappings, namely, we normalize by mappings that we already know are Möbius. This means that we consider the mapping $g = s_1 \circ f \circ s_2$, where f is our original Möbius mapping and s_1 and s_2 are similarities. Suppose first that f is such that $f(\infty) = \infty$. Inserting $d = \infty$ in the definition implies that

$$\frac{|a-c|}{|a-b|} = |a, b, c, \infty| = |f(a), f(b), f(c), \infty| = \frac{|f(a)-f(c)|}{|f(a)-f(b)|},$$

so f is a similarity. Otherwise there exists a finite point a such that $f(a) = \infty$. By an auxiliary similarity we may assume that $a = 0$ (i.e. we choose $s_2(x) = x + a$ above). Similarly, $f(\infty) = b \neq \infty$, and if we choose $s_1(x) = x - b$, then g is a Möbius mapping which swaps 0 and ∞. Using this in the equation gives

$$\frac{|b|}{|c|} = |\infty, b, c, 0| = |0, g(b), g(c), \infty| = \frac{|g(c)|}{|g(b)|}.$$

From this we see that $|x|\,|g(x)|$ is a constant. By another similarity we may assume that this constant equals 1, so that $|g(x)| = |x|^{-1}$ for every $x \in \mathbb{R}^n$. To get a grip of the non-radial action of g we use the equation inserting 0 and ∞ in other places:

$$\frac{|b-d|}{|d|} = |\infty, b, 0, d| = |0, g(b), \infty, g(d)| = \frac{|g(b)-g(d)|}{|g(b)|}$$

. Using the previous formula for $|g(b)|$ this gives

$$|g(b)-g(d)| = \frac{|b-d|}{|b|\,|d|}, \tag{2.1}$$

which is a central formula for calculating how a Möbius mapping affects distances. We can rewrite (2.1) as

$$\frac{|g(b)-g(d)|^2}{|g(b)|\,|g(d)|} = \frac{|b-d|^2}{|b|\,|d|}.$$

Using the cosine formula

$$|b-d|^2 = |b|^2 + |d|^2 - 2|b|\,|d| \cos\big(\widehat{b0d}\big),$$

where $\widehat{b0d}$ stands for the angle between the vectors $b-0$ and $d-0$, and similarly for $|g(b)-g(d)|^2$ we see that g preserves angles at the origin and lines through the origin. Thus, up to additional normalization by a reflection and/or a rotation, we see that $g(x) = x\,|x|^{-2}$.

A Möbius mapping which swaps ∞ and with a point of $\mathbb{R}^n$ and which maps every line through this point to itself is called an *inversion*. Note that every inversion is an involution, i.e. it is its own inverse. The point which is mapped to ∞ is called the center of inversion. We have shown that every inversion equals $x \mapsto x\,|x|^{-2}$, up to similarity. In particular, every Möbius mapping can be written as $s \circ i$, where s is a similarity and i is an inversion or the identity.

Now that we have identified the Möbius mappings we can proceed to show the following basic property: given two ordered triples of distinct points in $\overline{\mathbb{R}^n}$, (a, b, c) and (a', b', c'), there exists a Möbius map f with $f(a) = a'$, $f(b) = b'$ and $f(c) = c'$. It is clearly sufficient to show this claim in the case when a', b' and c' are the vertices of an equilateral triangle. Let us first find a point x such that

$$\frac{|a-c|}{|a-b|}\frac{|b-x|}{|c-x|} = \frac{|b-c|}{|a-b|}\frac{|a-x|}{|c-x|} = 1.$$

The easiest way to see that such a point x exists is to use an inversion i with center a. Then the equations to satisfy become

$$\frac{|i(b)-z|}{|i(c)-z|} = \frac{|i(b)-i(c)|}{|i(c)-z|} = 1.$$

We see that the first fraction describes a hyperplane which is the perpendicular bisector of the segment $[i(b), i(c)]$ and the second fraction the sphere with center $i(c)$ and radius $|i(b)-i(c)|$. Since these objects clearly intersect, we can find a suitable z, and then our x is given by $i(z)$. Let $\tilde{\imath}$ be an inversion with center x. The choice of x implies that

$$\frac{|\tilde{\imath}(a)-\tilde{\imath}(c)|}{|\tilde{\imath}(a)-\tilde{\imath}(b)|}=\frac{|\tilde{\imath}(b)-\tilde{\imath}(c)|}{|\tilde{\imath}(a)-\tilde{\imath}(b)|}=1,$$

so $\big(\tilde{\imath}(a),\tilde{\imath}(b),\tilde{\imath}(c)\big)$ are the vertices of an equilateral triangle. These can be mapped to the given points (a',b',c') by a similarity transform s, so our final Möbius map is then $s\circ\tilde{\imath}$.

3. The j_G metric

This section reproduces parts of the article [18] on isometries of some relative metrics. The term "relative metric" implies that the metric is evaluated in a proper subdomain of $\mathbb{R}^n$ relative to its boundary. More precisely, we want the metric to blow up towards the boundary of the domain, i.e., we want the boundary to be at infinity intrinsically.

Let $D\subsetneq\mathbb{R}^n$ be a domain containing the points x and y. The well-known *distance ratio metric* is defined by

$$j_D(x,y)=\log\left(1+\frac{|x-y|}{\min\{\delta(x),\delta(y)\}}\right),$$

where $\delta(\cdot)=\operatorname{dist}(\,\cdot\,,\partial D)$ denotes distance to the boundary. It was used, for instance, by Gehring and Osgood [11] to characterize uniform domains (namely, in such domains the j_D metric is quasiconvex). Note that this metric has sometimes been called simply "the relative metric", and will be used in this meaning.

To see how these metrics fit into a larger framework we recall the concept of an inner metric. Let d be a metric in D and γ be a path in D (i.e. a continuous mapping from an interval I to D). The length (or, more explicitly, d-length) of γ is defined as

$$d(\gamma)=\sup\sum_{i=1}^{k-1}d\big(\gamma(t_i),\gamma(t_{i+1})\big),$$

where the supremum is taken over k and all increasing sequences $(t_i)_{i=1}^k$ of points in I. Then the *inner* or *intrinsic metric* of d is defined by

$$\tilde{d}(x,y)=\inf_{\gamma}d(\gamma),$$

where the infimum is taken over all paths γ connecting x and y in D (note that this need not be finite, unless D is rectifiably connected). It is clear that $d(x,y)\le\tilde{d}(x,y)$ and that $d(\gamma)=\tilde{d}(\gamma)$ for any metric and path. The theory of length-metrics, including in particular intrinsic metrics, is presented e.g. in [5, 6].

Suppose now that $D\subset\mathbb{R}^n$ and d is a metric in D. If

$$\bar{d}(x)=\lim_{y\to x}\frac{d(x,y)}{|x-y|}$$

exists for all $x \in D$ and is continuous, then we can express the inner metric of d by

$$\tilde{d}(x,y) = \inf_\gamma \int_\gamma \bar{d}(z)\,|dz|,$$

where $|dz|$ represents integration with respect to d-arclength, and the infimum is taken over rectifiable curves with end points x and y. In this case $\tilde{d}$ is called a *conformal metric*. We easily see that the inner metric of j_D is the quasihyperbolic metric,

$$\tilde{j}_D(x,y) = k_D(x,y) = \inf_\gamma \int_\gamma \frac{|dz|}{d(z,\partial D)}.$$

Length-metrics are interesting from a geometric point of view, but for getting explicit estimates they are often of little use. The role of point-distance functions, like the j_D metric, is that they share features with their inner metrics, but are much more explicit.

In this paper we want to consider not only the j_D metric, but all metrics which resemble them in the very small and very large scale. The small scale equivalence implies that the metrics have the same inner metrics, whereas the large scale equivalence allows us to get a hold of the boundary behavior and thus start unraveling the isometry story.

Remark 3.1. Note that j_D is really families of metrics, namely for every domain D we have one metric. We will continue to use this convention when talking about this and other metrics in this paper.

Definition 3.2. We say that d is a *j-type metric* if the following three conditions hold on every domain $D \subsetneq \mathbb{R}^n$:

1. d_D is a metric on D.
2. For each $y \in D$ and for each sequence (x_i) with $j_D(x_i,y) \to 0$ we have
$$\lim_{i\to\infty} \frac{d_D(x_i,y)}{j_D(x_i,y)} = 1.$$
3. For each $y \in D$ and for each sequence (x_i) with $j_D(x_i,y) \to \infty$ we have
$$\lim_{i\to\infty} \big(d_D(x_i,y) - j_D(x_i,y)\big) = 0.$$

The fact that y can be any interior point in (2) means that being a j-type metric is quite a strong condition; for instance, if d and $f \circ d$ are j-type metrics, then $f = \mathrm{id}$ (Corollary 3.12).

It seems to be quite difficult to construct other natural metrics of j-type. The main purpose of our more abstract treatment is to highlight the features that are crucial, which in turn indicates that these techniques might be relevant also for handling the isometries of the corresponding inner metrics.

In this part we characterize the isometries of j-type metrics. We start by collecting some basic properties of these metrics. In Section 3.1 we solve the isometry problem for j-type metrics using a boundary rigidity result, and in Section 3.2 we list some additional properties whose proofs can be found in [18].

The following result is more or less restatements of the definition. However, it directly implies that we may restrict our focus very much without losing any isometries.

Proposition 3.3. *If d is a j-type, then*

$$\lim_{y\to\partial D\setminus\{\infty\}}\left(d_D(x,y)-\log\frac{|x-y|}{\delta(y)}\right)=0 \qquad \text{and} \qquad \lim_{y\to x}\frac{d_D(x,y)}{|x-y|}=\frac{1}{\delta(x)}$$

for every $x\in D$. The inner metric of a j-type metrics is the quasihyperbolic metric k_D. In particular, every isometry of a j-type metric is an isometry of k_D.

Every isometry of k_D is a conformal mapping [29, Theorem 2.6]. Hence we conclude:

Corollary 3.4. *Every isometry of a j or δ-type metric is conformal. In particular, if $n\geq 3$, then such an isometry is Möbius.*

We plunge right into the main result of this section, a characterization of the isometries of j-type metrics. In Section 3.2 we derive some miscellaneous results, which give a clearer picture of j-type metrics.

3.1. Isometries of j-type metrics. The proof of the following theorem is partly based on ideas from [17]. Incidentally, it is possible to give a much simpler proof for the particular case of the j_D metric itself, since in this case we can cancel the logarithm and the 1+ terms. Thus f is a j_D isometry if and only if

$$\frac{|x-y|}{\min\{\delta(x),\delta(y)\}}=\frac{|f(x)-f(y)|}{\min\{\delta'(f(x)),\delta'(f(y))\}},$$

where, as usual, δ' denotes the distance to the boundary in the image domain $f(D)$. The reader is challenged to find the very short argument which shows that this implies that the isometry is a similarity.

In the general case of j-type metrics we have less information about the metric, so we have to look at what happens at the boundary. In this case we can nevertheless prove the following theorem, whose proof is reproduced from [18].

Theorem 3.5. *Let d be a j-type metric, $D\subsetneq\mathbb{R}^n$ and $f\colon D\to\mathbb{R}^n$ be a d-isometry. Then either*

1. *f is a similarity, or*
2. *$D=\mathbb{R}^n\setminus\{a\}$ and, up to similarity, f is an inversion in a sphere centered at a.*

Proof. Denote $D'=f(D)$ and $\delta'(x)=d(x,\partial f(D))$. Fix $z\in\partial D\setminus\{\infty\}$ and let (z_i) be a sequence of points in D tending to z. We first assume that there exists a subsequence, which we also denote by (z_i), such that $\big(f(z_i)\big)$ converges to some point $w_1 \in \mathbb{R}^n$. Since d is a j-type metric we see, using Proposition 3.3 for the third equality, that for every $x \in D$ we have that

$$\begin{aligned}
0 &= \lim_{i\to\infty} \big(d_{D'}(f(x), f(z_i)) - d_D(x, z_i)\big) \\
&= \lim_{i\to\infty} \Big(d_{D'}(f(x), f(z_i)) - \log \frac{1}{\delta'(f(z_i))}\Big) - \lim_{i\to\infty} \Big(d_D(x, z_i) - \log \frac{1}{\delta(z_i)}\Big) + \\
&\qquad\qquad + \lim_{i\to\infty} \log \frac{\delta(z_i)}{\delta'(f(z_i))} \\
&= \log \frac{|f(x) - w_1|}{|x - z|} + \lim_{i\to\infty} \log \frac{\delta(z_i)}{\delta'(f(z_i))}.
\end{aligned}$$

Taking exponentials gives

$$\lim_{i\to\infty} \frac{\delta'(f(z_i))}{\delta(z_i)} = \frac{|f(x) - w_1|}{|x - z|} < \infty. \tag{3.6}$$

Suppose now that $(\hat{z}_i)$ is a second sequence of points in D tending to z, but that this time $f(\hat{z}_i) \to w_2 \in \overline{\mathbb{R}^n} \setminus \{w_1\}$. Using $x = \hat{z}_j$ for every $j = 1, 2, \ldots$ in (3.6) gives

$$\lim_{i\to\infty} \frac{\delta'(f(z_i))}{\delta(z_i)} = \frac{|f(\hat{z}_j) - w_1|}{|\hat{z}_j - z|} \to \infty$$

as $j \to \infty$, which is a contradiction. In other words, $f(\hat{z}_i) \to w_1$ for every sequence of points $(\hat{z}_i) \to z$, so we may extend f continuously to $\overline{D}$ by defining $f(z) = \lim_{i\to\infty} f(z_i)$. Therefore we conclude from (3.6), since the left-hand side of this equation does not depend on x, that

$$|f(x) - f(z)| = h_f(z)|x - z|$$

for some function $h_f \colon \partial D \to (0, \infty)$. This means that for $z, w \in \partial D$ we have

$$h_f(z)|w - z| = |f(w) - f(z)| = h_f(w)|w - z|,$$

so h_f is in fact a constant. Therefore f acts as a similarity, say g, on the boundary. We then extend f to all of $\mathbb{R}^n$ by setting $f(x) = g(x)$ outside the original domain of definition. Then it is clear that

$$|f(x) - f(z)| = h_f|x - z| \tag{3.7}$$

for every point $x \in \mathbb{R}^n$, i.e. the sphere $S^{n-1}(z, r)$ is mapped to $S^{n-1}(f(z), h_f r)$. This clearly implies that the conformal mapping is Möbius, and a Möbius mapping satisfying (3.7) is a similarity.

We still have one assumption to consider. In the beginning of the proof we assumed that we can find a boundary point z and a sequence (z_i) of points in D tending to z such that $f(z_i)$ tends to a finite limit. So we suppose now that no such sequence can be found, i.e. that for every sequence (z_i) of points in D tending to a boundary point z the sequence $\big(f(z_i)\big)$ tends to ∞. As before we conclude that

$$\begin{aligned}
0 &= \lim_{i\to\infty} \Big(d_{D'}(f(x), f(z_i)) - d_D(x, z_i) \Big) \\
&= \lim_{i\to\infty} \Big(j_{D'}(f(x), f(z_i)) - j_D(x, z_i) \Big) \\
&= \lim_{i\to\infty} \log \left(\frac{|f(x) - f(z_i)|}{\min\{\delta'(f(z_i)), \delta'(f(x))\}} \frac{\delta(z_i)}{|x - z|} \right).
\end{aligned}$$

So it follows that

$$\lim_{i\to\infty} \frac{|f(x) - f(z_i)|\delta(z_i)}{\min\{\delta'(f(z_i)), \delta'(f(x))\}} = |x - z|.$$

Since $f(z_i) \to \infty$, we see that we can replace $|f(x) - f(z_i)|$ by $|f(z_i)|$ in the above formula. Since the right-hand-side depends on x (which lies in an open set) we see that the left-hand-side must do so, too, hence we have to choose the second term in the minimum. Taking this into account we have

$$g_f(z) = \lim_{i\to\infty} |f(z_i)|\, \delta(z_i) = |x - z|\, \delta'(f(x)),$$

where $g_f \colon \partial D \to (0, \infty)$. Suppose that D has at least two finite boundary points, and let $a, b \in \partial D$ be such that the open segment (a, b) is contained in D. Now if we first consider x (in the previous equation) to be the mid-point x of (a, b), then we conclude that

$$g_f(a) = |x - a|\, \delta'(f(x)) = |x - b|\, \delta'(f(x)) = g_f(b).$$

But if we take some other point on the segment, then we get $g_f(a) \neq g_f(b)$, a contradiction. So only the case when D has a single boundary point remains to consider. Then we have

$$\lim_{i\to\infty} |f(z_i)|\, |z_i - a| = |x - a|\, |f(x) - b|$$

(for $D = \mathbb{R}^n \setminus \{a\}$ and $D' = \mathbb{R}^n \setminus \{b\}$) and we directly see that $x \mapsto f(x) + b - a$ is an inversion, which concludes the proof. ■

Corollary 3.8. *Let d be a similarity invariant j-type metric and let $D \subsetneq \mathbb{R}^n$. Then $f \colon D \to \mathbb{R}^n$ is a d-isometry if and only if*

1. *f is a similarity, or*
2. *$D = \mathbb{R}^n \setminus \{a\}$ and, up to similarity, f is the inversion in a sphere centered at a.*

Proof. The previous proposition established that every d-isometry is of the given kind. If f is a similarity, then it is an isometry by assumption. So it remains (after normalization) to consider the case $D = \mathbb{R}^n \setminus \{0\}$. In this case we see that similarity invariance implies that $d_D(x, y)$ depends only on $\max\left\{\frac{|x|}{|y|}, \frac{|y|}{|x|}\right\}$ and the angle $\widehat{x0y}$. On the other hand, an inversion in a sphere about the origin swaps $|x|/|y|$ and $|y|/|x|$ and leaves $\widehat{x0y}$ invariant, so we see that it is an isometry. ■

3.2. Other properties of j-type metrics. We said before that every j-type metric has an upper bound in terms of the quasihyperbolic metric. Surprisingly, it is also possible to get a universal lower bound by a metric, the so-called *half-apollonian metric* [19]. For a domain $D \subsetneq \mathbb{R}^n$ this metric is defined by

$$\eta_D(x,y) = \sup_{z\in\partial D} \left| \log \frac{|x-z|}{|y-z|} \right|.$$

The metric η_D is similarity invariant, and every Möbius mapping is bilipschitz.

The proofs of the following results can be found in [18].

Proposition 3.9. *For every j-type metric d and every $D \subsetneq \mathbb{R}^n$ we have $d_D \ge \eta_D$.*

Using the previous proposition and the quasihyperbolic upper bound we can squeeze in j-type metrics to get the exact value on some subset of the domain:

Corollary 3.10. *Let $w \in D$ and $z \in \partial D \cap S^{n-1}(w, \delta(w))$. Then $d_D(x,y) = j_D(x,y)$ for every $x, y \in [w, z)$.*

The following is easily checked by a direct computation using the definition of the j-metric, but also follows from Corollary 3.10 and the fact that line segments are also geodesic rays for the η_D-metric ([19, Example 3.4]).

Corollary 3.11. *Let $w \in D$ and $z \in \partial D \cap S^{n-1}(w, \delta(w))$. Then $[w, z)$ is a geodesic ray for the j-type metric d, i.e. for every $x, \xi, y \in [w, z)$ in this order we have*

$$d_D(x,y) = d_D(x,\xi) + d_D(\xi,y).$$

In general, if we have a metric d and a subadditive function $f\colon [0,\infty) \to [0,\infty)$ for which $f(x) = 0$ if and only if $x = 0$, then $f \circ d$ is also a metric. It turns out that the conditions for begin a j-type metric are so rigid, that this transformation is never possible in this context:

Corollary 3.12. *Let d be a j-type metric and $f_D\colon [0,\infty) \to [0,\infty)$ be a family of arbitrary functions. If $f \circ d$ is a j-type metric, then $f_D = \mathrm{id}$ for all relevant D.*

4. The quasihyperbolic metric

The remainder of this article is reproduced with minor modifications from [15].

Let $D \subsetneq \mathbb{R}^2$ be an open set and denote $\delta(x) = d(x, \partial D)$, the distance to the boundary. The quasihyperbolic metric in D is the conformal metric with the density $\delta(x)^{-1}$, in other words, the metric is given by

$$k_D(x,y) = \inf_\gamma \int_\gamma \frac{ds(z)}{\delta(z)},$$

where the infimum is taken over paths γ connecting x and y in D and ds represents integration with respect to arc-length.

The quasihyperbolic metric was first introduced in the seventies, and since then it has found innumerable applications, especially in the theory of quasiconformal mappings, see, e.g. [11, 12, 22, 28, 29]; new connections are still being made, for instance P. Jones and S. Smirnov [24] recently gave a criterion for removability of a set in the domain of definition of a Sobolev space in terms of the integrability of the quasihyperbolic metric, see also [25], and Z. Balogh and S. Buckley [1] used the metric in a geometric characterization of Gromov hyperbolic spaces.

Despite the prominence of the quasihyperbolic metric, there have been almost no investigations of its geometry. Three exceptions are the papers by G. Martin [28] and Martin and B. Osgood [29], the second of which was the main motivation for the approach presented in this paper, and the thesis by H. Lindén [27]. Part of the reason for this lack of geometrical investigations is probably that the density of the quasihyperbolic metric is not differentiable in the entire domain, which places the metric outside the standard framework of Riemanian metrics.

At least two modifications of the quasihyperbolic metric have been proposed which do not suffer from this problem. J. Ferrand [10] suggested replacing the density δ^{-1} by

$$\sigma_D(x) = \sup_{a,b\in\partial D} \frac{|a-b|}{|a-x|\,|b-x|}.$$

Note that $\delta(x)^{-1} \le \sigma_D(x) \le 2\delta(x)^{-1}$, so the Ferrand metric and the quasihyperbolic metric are bilipschitz equivalent. Moreover, the Ferrand metric is Möbius invariant, whereas the quasihyperbolic metric is only Möbius quasi-invariant. A second variant was proposed more recently by R. Kulkarni and U. Pinkall [26], see also [23]. The K–P metric is defined by the density

$$\mu_D(x) = \inf\Big\{ \frac{2r}{(r-|x-z|)^2} \colon x \in B(z,r) \subset D \Big\}.$$

Equivalently, the infimum is taken over the hyperbolic densities of x in balls contained in D. This density satisfies the same estimate as Ferrand's density, i.e. $\delta(x)^{-1} \le \mu_D(x) \le 2\delta(x)^{-1}$, and the K–P metric is also Möbius invariant. Although the Ferrand and K–P metrics are in some sense better behaved than the quasihyperbolic metric, they suffer from the short-coming that it is very difficult to get a grip even of the density, even in simple domains.

Despite this, D. Herron, Z. Ibragimov and D. Minda [21] recently managed to solve the isometry problem of the K–P metric in most cases. By the isometry problem of the metric d we mean characterizing mappings $f\colon D \to \mathbb{R}^2$ with

$$d_D(x,y) = d_{f(D)}(f(x), f(y))$$

for all $x, y \in D$. Notice that in some sense we are here dealing with two different metrics, due to the dependence on the domain. Hence the usual way of approaching the isometry problem is by looking at some intrinsic features of the metric which are then preserved under the isometry. Since irregularities (e.g. cusps) in the domain often lead to more distinctive features, this implies that the problem is often easier for more complicated domains.

The work by Herron, Ibragimov and Minda [21] bears out this heuristic – they were able to show that all isometries of the K–P metric are Möbius mappings except in simply and doubly connected domains. Their proof is based on studying the curvature of the metric. For the quasihyperbolic metric, formulae for the curvature were worked out already in [29] (see Section 4.2, below), and were used in that paper to prove that all the isometries of the disc are similarity mappings. These will be our main tool in this paper. The other source of the ideas used below are the papers [16, 17, 18, 19] on isometries of some other similarity and Möbius invariant metrics.

There are three steps in characterizing quasihyperbolic isometries:

1. show that they are conformal;
2. show that they are Möbius; and
3. show that they are similarities.

The first step has been carried out by Martin and Osgood [29, Theorem 2.6] for completely arbitrary domains, so there is no more work to do there. In Section 4.3 we will use the results from [29] on the curvature of the quasihyperbolic metric, and some new ideas to prove that the conformal isometries are Möbius (second step). For this we need to assume that the boundary of the domain is at least C^3-smooth. In Section 4.1 we will work on the third step – we show that Möbius isometries are similarities provided the boundary is C^1. In Section 4.2 we study the Gaussian curvature of the quasihyperbolic metric, and the gradient of the curvature.

Additional notation. We employ some additional conventions in this section: We tacitly identify $\mathbb{R}^2$ with $\mathbb{C}$, and speak about real and imaginary axes, etc. We will often work with a mapping $f\colon D \to \mathbb{R}^2$. In such cases we will use a prime to denote quantities on the image side, e.g. $x' = f(x)$, $D' = f(D)$ and $\delta'(x) = d(x, \partial D')$, and so on.

4.1. Isometries which are Möbius. Let D be a domain and $\zeta \in \partial D$. We say that ζ is *circularly accessible*, if there exists a disc $B \subset D$ such that $\zeta \in \partial B$.

Lemma 4.1. *Let $D \subsetneq \mathbb{R}^2$ be a Jordan domain with circularly accessible boundary, and let $f\colon D \to \mathbb{R}^2$ be a quasihyperbolic isometry which is also Möbius. Then, up to composition by similarity mappings, f is the identity or the inversion in a circle centered at a boundary point.*

Proof. Assume that f is not a similarity. Since f is a Möbius map, it is an inversion, up to similarities, which are always isometries of the quasihyperbolic metric. Thus it suffices to consider the case when f is an inversion in a unit sphere. Let us denote the center of this sphere by w.

Suppose first that $w \notin \overline{D}$ and let $\zeta \in \partial D$ be the closest boundary point to w. For simplicity we normalize the situation so that ζ lies on the positive real axis and $w = 0$. Since ζ is circularly accessible, we find a disc $B(z, r) \subset D$ which contains ζ in its closure. Since ζ is the closest boundary point to w, we see that

z has to lie on the positive real axis, as well. Let x and y be points satisfying $\zeta < x < y \le \frac{\zeta(\zeta+2r)}{\zeta+r}$. The right-hand inequality ensures that ζ is the closest boundary point to $[x, y]$, and that ζ' is the closest boundary point to $[x', y']$. Thus we find that

$$k_D(x, y) = \log \frac{|x-\zeta|}{|y-\zeta|} \quad \text{and} \quad k_{D'}(x', y') = \log \frac{|x'-\zeta'|}{|y'-\zeta'|}.$$

Since f is the inversion in the unit sphere, we have

$$|x' - \zeta'| = \frac{|x-\zeta|}{|x|\,|\zeta|},$$

and similarly for y. Then the equation $\exp k_D(x, y) = \exp k_{D'}(x', y')$ gives us

$$\frac{|x-\zeta|}{|y-\zeta|} = \frac{|x-\zeta|}{|x|\,|\zeta|}\frac{|y|\,|\zeta|}{|y-\zeta|},$$

i.e. $|x| = |y|$. This contradiction shows that $w \in \overline{D}$. Since f maps D into $\mathbb{R}^2$, it is clear that $w \notin D$, so it follows that w is a boundary point. ∎

We call D a C^k *domain*, if ∂D is locally the graph of a C^k function. Note that if D is a C^1 domain, then certainly every boundary point is circularly accessible.

Proposition 4.2. *Let $D \subsetneq \mathbb{R}^2$ be a C^1 domain, and let $f\colon D \to \mathbb{R}^2$ be a quasihyperbolic isometry which is also Möbius. If D is not a half-plane, then f is a similarity.*

Proof. We assume that f is not a similarity map. By the previous lemma we see that there is no loss of generality in considering only the case when f is the inversion centered at a boundary point. For simplicity of exposition, we normalize so that the origin is this center.

Let ζ be a boundary point of D distinct from 0 and let u be the inward pointing unit normal at ζ. For all sufficiently small $t > 0$, the point $x_t = \zeta + tu$ lies in D and its closest boundary point is ζ. For such $s < t$, we have

$$k_D(x_t, x_s) = \log \frac{t}{s}.$$

To estimate the distance of the image points, we use the inequality

$$j_{D'}(x', y') = \log\left(1 + \frac{|x'-y'|}{\min\{\delta'(x'), \delta'(y')\}}\right) \le k_{D'}(x', y'),$$

which is always valid (since $k_{D'}$ is the inner metric of $j_{D'}$, e.g. [12, Lemma 2.1]). We also need the formula

$$|x' - y'| = \frac{|x-y|}{|x|\,|y|}$$

for the length distortion of an inversion. Using these facts and the estimate $\delta'(x') \le |x' - \zeta'|$, we derive the inequality

$$\begin{aligned} k_{D'}(x', y') &\ge \log\Big(1 + \frac{|x' - y'|}{\min\{\delta'(x'), \delta'(y')\}}\Big) \\ &\ge \log\Big(1 + \frac{|x - y|/(|x|\,|y|)}{\min\{|x' - \zeta'|, |y' - \zeta'|\}}\Big) \\ &= \log\Big(1 + \frac{|x - y|\,|\zeta|}{|x|\,|y|\,\min\{|x - \zeta|/|x|, |y - \zeta|/|y|\}}\Big) \\ &= \log\Big(1 + \frac{|x - y|\,|\zeta|}{\min\{|y|\,|x - \zeta|, |x|\,|y - \zeta|\}}\Big). \end{aligned}$$

Applying this inequality to the points x_t and x_s as defined before, we have

$$k_{D'}(x'_t, x'_s) \ge \log\Big(1 + \frac{(t - s)\,|\zeta|}{\min\{t\,|x_s|, s\,|x_t|\}}\Big).$$

Let us choose $t = 2s$. Since $|x_{2s}|$ and $|x_s|$ both tend to $|\zeta|$ as $s \to 0$, we see that the second term in the minimum is smaller. Since the inversion is supposed to be an isometry, we can use the formula for $k_D(x_t, x_s)$ from before with the previous inequality to conclude that

$$\log\frac{2s}{s} \ge \log\Big(1 + \frac{(2s - s)\,|\zeta|}{s\,|x_{2s}|}\Big).$$

Taking the exponential function gives $|x_{2s}| \ge |\zeta|$. Since $x_s = \zeta + su$, this implies that $\langle \zeta - 0, u\rangle \ge 0$ as $s \to 0$, where $\langle , \rangle$ denotes the scalar product.

Applying the same argument, but starting with points on the image side, we conclude that the opposite inequality is also valid. (There is actually a slight asymmetry here: the domain D' need not have circularly accessible boundary at the origin. However, it is clear that this does not affect the argument so far.) Thus it follows that $\langle \zeta - 0, u\rangle = 0$ for all boundary points. But since the boundary is assumed to be C^1, this implies that the domain is a half-plane. ∎

From [29, Theorem 2.8] we know that if $f\colon D \to \mathbb{R}^2$ is a quasihyperbolic isometry, then f is conformal in D. In dimensions three and higher every conformal mapping is Möbius. It is easy to see that the proofs in this section work also in the higher dimensional case. Therefore, we have proved the following result:

Corollary 4.3. *Let D be a C^1 domains in $\mathbb{R}^n$, $n \ge 3$, which is not a half-space. Then every quasihyperbolic isometry is a similarity mapping.*

Example 4.4. Note that if we do not assume C^1 boundary, then there are some further domains with non-trivial isometries: the punctured planes $\mathbb{R}^2 \setminus \{a\}$ and sector domains (i.e. domains whose boundary consists of two rays). In both cases inversions centered at the distinguished boundary point (a or the vertex of the sector). The previous proposition strongly suggests that these are all the examples.

4.2. Curvature of the quasihyperbolic metric. Let D be a domain in $\mathbb{R}^2$. We call a disc $B \subset D$ maximal, if it is not contained in any other disc contained in D. The set consisting of the centers of all maximal discs in D is called the *medial axis* of D and denoted by $\mathrm{MA}(D)$. The medial axis and differentiability properties of the distance-to-the-boundary function have been studied e.g. in [7, 8, 9].

In a general domain the Gaussian curvature of the quasihyperbolic metric is not defined, since the distance-to-the-boundary function is not C^2. M. Heins [20] considered this situation for a quite general class of metric, and defined the notions of upper and lower curvature. Martin and Osgood worked with these curvatures in the context of the quasihyperbolic metric, see [29, Section 3] for details. However, if our domain is sufficiently regular (say C^2), and we are considering points not on the medial axis, then the upper and lower curvature agree, and define the curvature. In this case the curvature of k_D is given by

$$\mathcal{K}_D(z) = -\delta(z)^2 \triangle \log \delta(z),$$

[20, (1.3)] or [29, (3.1)]. On the medial axis this formula does not make sense, but the upper and lower curvatures still agree, and both equal $-\infty$, by [29, Corollary 3.12].

The next lemma is a specialization of Lemma 3.5, [29] to the case there the upper and lower curvatures agree.

Lemma 4.5 (Lemma 3.5, [29])**.** *Let G and $\tilde{G}$ be C^2 domains such that $B(z,r) \subset G \cap \tilde{G}$ and $\zeta \in (\partial G) \cap (\partial \tilde{G}) \cap (\partial B(z,r))$. If there is a neighborhood U of ζ such that $G \cap U \subset \tilde{G} \cap U$ and $d(z, \partial \tilde{G} \setminus U) > d(z, \partial \tilde{G})$, then $\mathcal{K}_G(z) \le \mathcal{K}_{\tilde{G}}(z)$.*

Using this lemma we can derive the following very plausible statement, which says that the Gaussian curvature of the quasihyperbolic metric depends only on the curvature of the boundary at the closest boundary point. We sill need some more notation.

Let B be a disc with $\zeta \in (\partial B) \cap (\partial D)$. Then we call B the *osculating* disc at ζ if ∂B and ∂D have second order contact at ζ. Let D be at least a C^2 domain. Then there exists an osculating disc at every boundary point ζ. If this disc has radius r, then we define R_ζ to be r if the disc lies in the direction of the interior of D, and $-r$ otherwise. Note that the function $\zeta \mapsto 1/R_\zeta$ is C^{k-2} in a C^k domain, $k \ge 2$.

Proposition 4.6. *Let $D \subsetneq \mathbb{R}^2$ be a C^2 domain and $z \in D \setminus \mathrm{MA}(D)$ have closest boundary point $\zeta \in \partial D$. Then*

$$\mathcal{K}_D(z) = -\frac{R_\zeta}{R_\zeta - \delta(z)} = -\frac{1}{1 - \delta(z)/R_\zeta}.$$

If z lies on the medial axis, then $\mathcal{K}_D(z) = -\infty$.

Proof. The medial axis consists of points equidistant to two or more nearest boundary points, and of centers of osculating circles. For the former, the claim that $\mathcal{K}_D(z) = -\infty$ follows from [29, Corollary 3.12]. So we assume that z has a unique nearest boundary point, ζ.

We suppose further that $R_\zeta > 0$, the other case begin similar. Let $B(w, R_\zeta)$ be the osculating disc at ζ. We define

$$B_t = B(w + \tfrac{w-\zeta}{R_\zeta} t, R_\zeta + t),$$

and note that ∂B_t contains ζ for all $t > -R_\zeta$. We have the formula

$$\mathcal{K}_{B(0,r)}(x) = -\frac{r}{|x|} = -\frac{r}{r - d(x, \partial B(0,r))}$$

for the curvature of the quasihyperbolic metric in a ball [29, Lemma 3.7], so we can calculate $\mathcal{K}_{B_t}(z)$ explicitly.

Using the previous lemma with $G = D$ and $\tilde{G} = B_t$ for $t > 0$ gives $\mathcal{K}_D(z) \le \mathcal{K}_{B_t}(z)$. If z is the center of B_0, then right-hand-side of this inequality tends to $-\infty$ as $t \to 0$, which completes the proof of the claim regarding the medial axis. So we assume that z is not the center of B_0, and then we can apply the Lemma 4.5 with $G = B_t$ for $t < 0$ (sufficiently close to 0) and $\tilde{G} = D$ to get $\mathcal{K}_{B_t}(z) \le \mathcal{K}_D(z)$. Thus we have

$$\mathcal{K}_{B_{-t}}(z) \le \mathcal{K}_D(z) \le \mathcal{K}_{B_t}(z)$$

for small $t > 0$. Since $\mathcal{K}_{B_t}$ is continuous in t, we get $\mathcal{K}_D(z) = \mathcal{K}_{B_0}(z)$ as we let $t \to 0$. The proof is completed by applying the aforementioned formula for the curvature to the ball $B_0 = B(w, R_\zeta)$. ∎

Let $f\colon D \to \mathbb{R}^2$ be a C^1 mapping. By ∇f we denote the gradient of f, i.e. the vector $(\partial_1 f, \partial_2 f)$, and by $\tilde{\nabla} f(z)$ we denote $\delta(z)\nabla f(z)$. The reason for multiplying by $\delta(x)$ is that

$$\delta(y) = \lim_{x \to y} \frac{|x-y|}{k_D(x,y)},$$

so that the $\tilde{\nabla}$ operator is more natural in the setting where the quasihyperbolic but not the Euclidean distance is preserved (see (4.9), below).

We next present an explicit formula for $\tilde{\nabla}\mathcal{K}_D$. For this need a mapping which associates to every point in $D \setminus \mathrm{MA}(D)$ its closest boundary point. We call this mapping $\zeta = \zeta(z)$.

Lemma 4.7. *Let $D \subsetneq \mathbb{R}^2$ be a C^3 domain. Then*

$$\tilde{\nabla}\mathcal{K}_D(z) = (\mathcal{K}_D(z) + 1)\big[\mathcal{K}_D(z)\nabla\delta(z) - (\mathcal{K}_D(z) + 1)\nabla R_{\zeta(z)}\big]$$

for every z off the medial axis, where all differentiation is with respect to the variable z.

Proof. We use the formula from Proposition 4.6. Thus

$$\nabla\mathcal{K}_D(z) = -\nabla\frac{1}{1 - \delta(z)/R_\zeta} = \mathcal{K}_D(z)^2 \nabla\frac{\delta(z)}{R_\zeta} = \frac{\mathcal{K}_D(z)^2}{R_\zeta^2}(R_\zeta\nabla\delta(z) - \delta(z)\nabla R_\zeta),$$

where we understand ζ as a function of z. Note that R_ζ and δ are C^1, since D is C^3 and we are not on the medial axis. From Proposition 4.6 we also get

$$\frac{\delta(z)}{R_\zeta} = \frac{\mathcal{K}_D(z)+1}{\mathcal{K}_D(z)}.$$

Thus we continue the equation by

$$\begin{aligned}\tilde{\nabla}\mathcal{K}_D(z) &= \mathcal{K}_D(z)^2\frac{\delta(z)}{R_\zeta}\Big(\nabla\delta(z) - \frac{\delta(z)}{R_\zeta}\nabla R_\zeta\Big)\\ &= (\mathcal{K}_D(z)+1)\big(\mathcal{K}_D(z)\nabla\delta(z) - (\mathcal{K}_D(z)+1)\nabla R_\zeta\big). \quad \blacksquare\end{aligned}$$

We next show that $|\tilde{\nabla}\mathcal{K}|$ is an intrinsic quantity of the quasihyperbolic metric.

Lemma 4.8. *Let D be a C^3 domain. If $f\colon D \to \mathbb{R}^2$ is a quasihyperbolic isometry, then $|\tilde{\nabla}\mathcal{K}_D(z)| = |\tilde{\nabla}\mathcal{K}_{f(D)}(f(z))|$ for every $z \in D$.*

Proof. We know that f is conformal. For a unit vector u we find that

$$\begin{aligned}\big\langle\tilde{\nabla}\mathcal{K}_D(z), u\big\rangle &= \lim_{\varepsilon\to 0}\frac{\mathcal{K}_D(z+\varepsilon u) - \mathcal{K}_D(z)}{k_D(z+\varepsilon u, z)}\\ &= \lim_{\varepsilon\to 0}\frac{\mathcal{K}_{f(D)}(f(z+\varepsilon u)) - \mathcal{K}_{f(D)}(f(z))}{k_{f(D)}(f(z+\varepsilon u), f(z))}.\end{aligned} \tag{4.9}$$

Next we note that $f(z+\varepsilon u) = f(z)+\varepsilon f'(z)u + O(\varepsilon^2)$. Here $f'(z)u$ is understood as complex multiplication. Let us define another unit vector $\tilde{u} = \frac{f'(z)}{|f'(z)|}u$. Then we continue the previous equation by

$$\begin{aligned}\big\langle\tilde{\nabla}\mathcal{K}_D(z), u\big\rangle &= \lim_{\varepsilon\to 0}\frac{\mathcal{K}_{f(D)}(f(z)+\varepsilon f'(z)u) - \mathcal{K}_{f(D)}(f(z))}{k_{f(D)}(f(z)+\varepsilon f'(z)u, f(z))}\\ &= \lim_{\varepsilon\to 0}\frac{\varepsilon|f'(z)|\langle\nabla\mathcal{K}_{f(D)}(f(z)), \tilde{u}\rangle}{\varepsilon|f'(z)|\delta'(f(z))^{-1}}\\ &= \big\langle\tilde{\nabla}\mathcal{K}_{f(D)}(f(z)), \tilde{u}\big\rangle.\end{aligned}$$

Since u was an arbitrary unit vector, we see that $|\tilde{\nabla}\mathcal{K}_D(z)| = |\tilde{\nabla}\mathcal{K}_{f(D)}(f(z))|$. $\blacksquare$

4.3. Isometries of the quasihyperbolic metric. We know that similarities are always quasihyperbolic isometries, and we want to show that in most cases these are the only ones. In view of the results in Section 4.1, it suffices for us to show that a quasihyperbolic isometry is a Möbius mapping, so this will be what we aim at in the proofs of this section.

A curve γ in D is a (quasihyperbolic) geodesic if

$$k_D(x,y) = k_D(x,z) + k_D(z,y)$$

for all $x, z, y \in \gamma$ in this order. It is clear from this definition that geodesics are preserved by isometries. A geodesic ray is a geodesic which is isometric to $\mathbb{R}^+$. For every $z \in D$ we easily find one geodesic ray, namely $[z, \zeta(z))$, which also happens to be a Euclidean line segment. The idea is to show that this geodesic is somehow special (from a quasihyperbolic point-of-view), so that it would map to a geodesic ray of the same kind.

Lemma 4.10. *Let $D \subsetneq \mathbb{R}^2$ be a C^2 domain with a boundary point ξ such that $1/R_\xi = 0$. Then every isometry $f\colon D \to \mathbb{R}^2$ of the quasihyperbolic metric is Möbius.*

Proof. Let $B \subset D$ be a non-maximal disc whose boundary contains ξ and let z denote the center of B. By Proposition 4.6 we find that $\mathcal{K}_D \equiv -1$ on the segment $\gamma = [z, \xi)$. Thus $\mathcal{K}_{f(D)} \equiv -1$ on γ', so $1/R'_{\zeta'(z')} = 0$ for every point z' on this curve. We consider two cases: either $\zeta'(z')$ is just a single point for all $z' \in \gamma'$, or it sweeps out a non-degenerate subcurve of the boundary $\partial D'$ as z' varies over γ'. (There is no third possibility, since ζ' is a continuous function on γ'.) In the single-point case we see that γ' has to be a line segment, since the boundary does not have corners. In this case we find that

$$k_D(x,y) = \Big|\log \tfrac{|x-\xi|}{|y-\xi|}\Big| \quad \text{and} \quad k_{D'}(x',y') = \Big|\log \tfrac{|x'-\xi'|}{|y'-\xi'|}\Big|,$$

where ξ' is the closest boundary point to the every point on γ'. But this easily implies that f is Möbius on γ. Since f is conformal it follows by uniqueness of analytic extension that f is a Möbius mapping on all of D.

So we consider the second case, that $\zeta'(z')$ sweeps out a non-degenerate subcurve of the boundary $\partial D'$. Since the curvature of the boundary at all these points is zero, it follows that the piece of the boundary is a line segment, L'.

Let $U' \subset D'$ be an open set such that $(\partial U') \cap (\partial D') = L'$ and the nearest boundary point of every point in U' lies in L'. Then the geometry of the quasihyperbolic metric in U is the same as in a half-plane, in particular $\mathcal{K}_{D'} \equiv -1$ on U'. Then $\mathcal{K}_D \equiv -1$ on $U = f^{-1}(U')$, so it follows that $(\partial U) \cap (\partial D) = L$, for some line segment L. So it follows that $f|_U$ is the restriction of a quasihyperbolic isometry of the half-plane. But these are only the Möbius mappings. Then we again conclude from the uniqueness of analytic extension that f is a Möbius mapping on all of D. ∎

Let us call a domain strictly concave, if its complement is strictly convex.

Corollary 4.11. *Let $D \subsetneq \mathbb{R}^2$ be a C^2 domain which is not a half-plane, strictly convex or strictly concave. Then every quasihyperbolic isometry is a similarity mapping.*

Proof. Suppose that $1/R_\zeta \neq 0$ for all boundary points. Since $1/R_\zeta$ is continuous by assumption, this implies that it is either everywhere positive, or everywhere negative. In these cases we have a strictly convex and strictly concave domain, respectively, which was ruled out by assumption. So we find some point at which $1/R_\zeta = 0$. Then it follows from Lemma 4.10 that the isometry is Möbius and from Lemma 4.2 that it is a similarity. ∎

So we are left with only two types of domains that we cannot handle: strictly convex and strictly concave ones. As usual when working with isometries, the nicest domains turn out to be the most difficult. Unfortunately, we need to assume more regularity of the boundary in order to take care of these cases.

Theorem 4.12. *Let $D \subsetneq \mathbb{R}^2$ be a C^3 domain, which is not a half-plane. Then every isometry $f\colon D \to \mathbb{R}^2$ of the quasihyperbolic metric is a similarity mapping.*

Proof. In view of Corollary 4.11, we may restrict ourselves to the case when $\mathcal{K}_D(z) \neq -1$ for all $z \in D$. Let $z \in D \setminus \mathrm{MA}(D)$ and ζ be its nearest boundary point. We note that $\nabla\delta(z)$ and ∇R_ζ are perpendicular – first of all, $\nabla\delta(z)$ is parallel to $z-\zeta$; second, R_ζ is a constant in the direction of $z-\zeta$, since ζ is the closest boundary point to all points on this line (near z).

If D is bounded, then it is clear that R_ζ has a critical point. If D is unbounded, then we note that $1/R_\zeta$ cannot have any other limit than 0 at ∞ (although a limit need not exist, of course). Thus we see that R_ζ has a critical point in the unbounded case as well. Let ζ be a critical point of $\xi \mapsto R_\xi$ and fix a point $z \in D$ with $\mathcal{K}_D(z) \neq -\infty$ whose nearest boundary point is ζ. Of course, $\nabla R_\zeta = 0$ at the critical point ζ. Then it follows from Lemma 4.7 that

$$\tilde{\nabla}\mathcal{K}_D(z) = (\mathcal{K}_D(z)+1)\mathcal{K}_D(z)\nabla\delta(z).$$

Since the curvature is intrinsic to the metric, we have $\mathcal{K}_{D'}(z') = \mathcal{K}_D(z)$. Also, $|\tilde{\nabla}\mathcal{K}_{D'}(z')| = |\tilde{\nabla}\mathcal{K}_D(z)|$ by Lemma 4.8, so we have

$$\big|(\mathcal{K}_D(z)+1)\mathcal{K}_D(z)\nabla\delta(z)\big| = \big|(\mathcal{K}_D(z)+1)\big[\mathcal{K}_D(z)\nabla\delta'(z') - (\mathcal{K}_D(z)+1)\nabla R'_{\zeta'(z')}\big]\big|.$$

We know that $\mathcal{K}_D(z) \neq -1$ and that $\nabla\delta'(z')$ and $\nabla R'_{\zeta'(z')}$ are orthogonal. Thus the previous equation simplifies to

$$\big(\mathcal{K}_D(z)|\nabla\delta(z)|\big)^2 = \big(\mathcal{K}_D(z)|\nabla\delta'(z')|\big)^2 + \big((\mathcal{K}_D(z)+1)\big|\nabla R'_{\zeta'(z')}\big|\big)^2.$$

Since $|\nabla\delta| = 1$ off the medial axis for every domain, this equation implies that $\nabla R_{\zeta'} = 0$.

So for our point z, $\nabla\mathcal{K}_D(z)$ and $\nabla\mathcal{K}_{D'}(z')$ point to the nearest boundary point of z and z', respectively. Let $\gamma = [z,\zeta)$. Note that γ is a geodesic of the quasihyperbolic metric. Also, $\nabla\mathcal{K}_D(z)$ and γ are parallel at z. Now γ is mapped to some geodesic ray γ', and since f is a conformal mapping, γ' is parallel to $\nabla\mathcal{K}_{D'}(z')$ at z'. But $[z',\zeta')$ is a geodesic parallel to $\nabla\mathcal{K}_{D'}(z')$ at z', and since geodesics are unique (when the density is C^2, i.e. except possibly on the medial axis) we see that $\gamma' = [z',\zeta')$.

So we have shown that $f([z,\zeta)) = [z',\zeta')$. Moreover, we have

$$k_D(x,y) = \left|\log\tfrac{|x-\zeta|}{|y-\zeta|}\right| \quad \text{and} \quad k_{D'}(x',y') = \left|\log\tfrac{|x'-\zeta'|}{|y'-\zeta'|}\right|$$

for $x,y \in [z,\zeta)$. Thus we see that f is just a similarity on $[z,\zeta)$. But f is a conformal map, so this implies that f is a similarity in all of D. ∎

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