Quantum Information: Part I

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Quantum Information

- What is quantum information?
- Reply: Quantum information is information represented in quantum systems.
- What is information?
- What is a quantum system?

Information

- Information is difference of entropies
- What is entropy?

Entropy

$$S = k \cdot \log W,$$

where k is a constant, W is the number of microstates corresponding to a macroscopic state

Entropy

“Elementary” entropy $H(n) =$ number of elementary units (bits, trits, etc.) to coin n (uniform) conditions.

- Binary entropy: $\{1, 2\} \mapsto \{0, 1\}$,
- $\{1, 2, 3\} \mapsto \{0, 1, 00\}$,
- $\{1, 2, 3, 4\} \mapsto \{00, 01, 10, 11\}$, etc.
- $H_2(n) = \log_2 n = \frac{1}{\log 2} \log n$
- $H_3(n) = \frac{1}{\log 3} \log n$, etc.
- Measure of “uncertainty”

Entropy

Boltzmann: Identical particles with same internal condition indistinguishable.

- Let l be the number of particles, each having n potential conditions $\{1, 2, \ldots, n\}$, $l \gg n$.
- List the conditions of all particles: $c_1 c_2 \ldots c_l$, $c_i \in \{1, 2, \ldots, n\}$
- Assume condition i occurs k_i times, so $k_1 + \ldots + k_n = l$ and $p_i = \frac{k_i}{l}$ is the probability (frequency) of condition i

Entropy

There are

$$\frac{l!}{k_1! \dots k_n!}$$

such lists (strings of conditions)
Entropy:

$$K \log \frac{l!}{k_1! \dots k_n!}$$

Per particle:

$$\frac{K}{l} \log \frac{l!}{k_1! \dots k_n!}$$

Entropy

Stirling: $\log k! = k \log k - k + O(\log k)$, so

$$
\begin{aligned}
& \frac{K}{l} \log \frac{l!}{k_1! \dots k_n!} \\
= \quad & \frac{K}{l}(l \log l - l + O(\log l) \\
& - \sum_{i=1}^{n} (k_i \log k_i - k_i + O(\log k_i))) \\
= \quad & -K \sum_{i=1}^{n} p_i \log p_i + O(\frac{\log l}{l}).
\end{aligned}
$$

Entropy

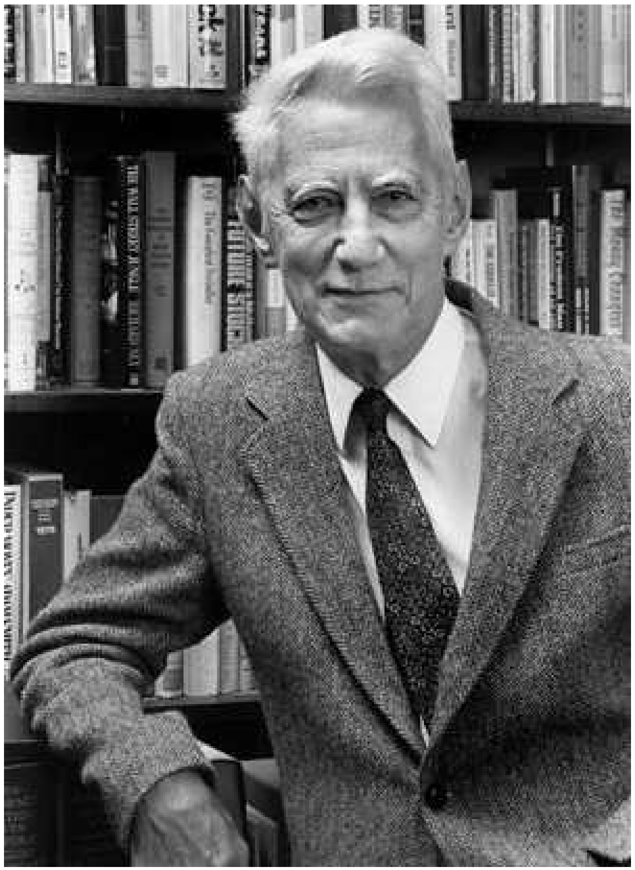
For a probability distribution $(p_1, \ldots, p_n)$ of events $\{e_1, \ldots, e_n\}$, define

$$H(p_1, \ldots, p_n) = -K \sum_{i=1}^{n} p_i \log p_i$$

For $p_1 = \ldots = p_n = \frac{1}{n}$

$$H(\frac{1}{n}, \ldots, \frac{1}{n}) = -K \cdot n \cdot \frac{1}{n} \log \frac{1}{n} = K \log n$$

Entropy



Claude Shannon (1916-2001)

- $H(p_1, \ldots, p_n)$ symmetric, continuous
- $H(\frac{1}{n}, \ldots, \frac{1}{n})$ non-negative, strictly increasing in n
- $H(p_1, \ldots, p_n) + p_n H(q_1, \ldots, q_m) = H(p_1, \ldots, p_{n-1}, p_n q_1, \ldots, p_n q_m)$

$$\Rightarrow H(p_1, \ldots, p_m) = -K \sum_{i=1}^{n} p_i \log p_i$$

Joint Entropy

Let

$$X = \{x_1, \ldots, x_n\}$$

be a random variable with distribution $p(x_1)$, ..., $p(x_n)$. Then

$$H(X) = -\sum_{i=1}^{n} p(x_i) \log p(x_i).$$

If also $Y = \{y_1, \ldots, y_m\}$ is a random variable, the *joint entropy* is

$$H(X,Y) = -\sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) \log p(x_i, y_j)$$

Conditional Entropy

$$H(X,Y) = -\sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i, y_j)\log p(x_i, y_j),$$

but if Y is known to assume value y_j, then

$$H(X \mid y_j) = -\sum_{i=1}^{n} p(x_i \mid y_j)\log p(x_i \mid y_j),$$

and

$$H(X \mid Y) = \sum_{j=1}^{m} p(y_j) H(X \mid y_j).$$

“Uncertainty of X when Y is known”

Conditional Entropy

Lemma: $H(X \mid Y) \leq H(X)$

Lemma: $H(X,Y) \leq H(X) + H(Y)$

Lemma: $H(X \mid Y) = H(X,Y) - H(Y)$

(Uncertainty of X when Y is known)

Conditional Entropy

Example: Team A wins with probability $\frac{1}{2}$, $X = \{\text{win}, \text{loss}\}$.
Then $H(X) = -(\frac{1}{2}\log_2\frac{1}{2} + \frac{1}{2}\log_2\frac{1}{2}) = 1$.

As a *home team*, A wins with $\frac{3}{4}$ probability, but as *visitor*, A wins only with $\frac{1}{3}$ probability.

$$\begin{aligned} H(X \mid h) &= -(\frac{3}{4}\log_2\frac{3}{4} + \frac{1}{4}\log_2\frac{1}{4}) = 0.811278\ldots, \\ H(X \mid v) &= -(\frac{1}{3}\log_2\frac{1}{3} + \frac{2}{3}\log_2\frac{2}{3}) = 0.918296\ldots \end{aligned}$$

Conditional Entropy

$$H(X \mid h) = -(\frac{3}{4}\log_2\frac{3}{4} + \frac{1}{4}\log_2\frac{1}{4}) = 0.811278\ldots,$$
$$H(X \mid v) = -(\frac{1}{2}\log_2\frac{1}{2} + \frac{1}{2}\log_2\frac{1}{2}) = 0.918296\ldots$$

Let $Y = \{0, 1\}$ be a fair coin toss for deciding if team A plays home. Then

$$H(X \mid Y) = \frac{1}{2}H(X \mid h) + \frac{1}{2}H(X \mid v) = 0.864787\ldots$$

Information

Mutual information of X and Y:

$$I(X:Y) = H(X) - H(X \mid Y)$$

$$\begin{aligned} I(X:Y) &= H(X) - H(X \mid Y) \\ &= H(X) - (H(X,Y) - H(Y)) \\ &= H(X) + H(Y) - H(X,Y) \\ &= I(Y:X) \end{aligned}$$

"Uncertainty of X minus uncertainty of X when Y known"

Previous example:

$$I(X:Y) = 1 - 0.864787\ldots = 0.135213\ldots$$

Quantum Information

John von Neumann (1903–1957)

- Quantum entropy by Gedanken Experiment (1927)
- Coincides with Shannon (and Boltzmann) entropy on classical systems

Mechanics

Newtonian equation of motion:

$$F = ma = m\frac{d}{dt}v = \frac{d}{dt}mv = \frac{d}{dt}p$$

Total energy:

$$H = \frac{1}{2}mv^2 + V(x) = \frac{p^2}{2m} - \int_{x_0}^{x} F(s)\, ds$$

Hamiltonian reformulation:

$$\frac{d}{dt}x = \frac{\partial}{\partial p}H, \quad \frac{d}{dt}p = -\frac{\partial}{\partial x}H$$

Mechanics

Classical:

$$\frac{d}{dt}x = \frac{\partial}{\partial p}H, \quad \frac{d}{dt}p = -\frac{\partial}{\partial x}H$$

Quantum:

$$\frac{\partial}{\partial t}\psi = -iH\psi,$$

where ψ is the *wave function*.

Wave Function

Max Born's interpretation:

$$|\psi(x,t)|^2$$

is the probability density of the particle position at time t:

$$\mathbb{P}(a \le x \le b) = \int_a^b |\psi(x,t)|^2 \, dx$$

On the other hand (omitting t):

$$\widehat{\psi}(p) = \mathcal{F}[\psi(x)](p) = \int_{-\infty}^{\infty} \psi(x) e^{-2\pi i x p} \, dx$$

is the probability density of the particle momentum.

Wave Function

On the other hand (omitting t):

$$\widehat{\psi}(p) = \mathcal{F}[\psi(x)](p) = \int_{-\infty}^{\infty} \psi(x) e^{-2\pi i x p} \, dx$$

is the probability density of the particle momentum:

$$\mathbb{P}(a \leq p \leq b) = \int_a^b \left| \widehat{\psi}(p) \right|^2 dp$$

ψ gives the complete characterization of the system at a fixed time

Finite Quantum Systems

- Nuclear spin
- Photon polarization

Wavefunction ψ defined on a finite set.

Formally,

$$\psi = \alpha_1 \boldsymbol{\psi}_1 + \alpha_2 \boldsymbol{\psi}_2 + \ldots + \alpha_n \boldsymbol{\psi}_n,$$

where $\{\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_n\}$ is an orthonormal basis of n-dimensional complex vector space.

For *mixed states*, representation must be generalized.

Formalism of Quantum Mechanics

John von Neumann (1903–1957)

- Hilbert space
- Linear mappings (operators)

Formalism

Paul Dirac (1902-1984)

Bra-ket notions

$\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle$, $|\boldsymbol{y}\rangle$, $\langle \boldsymbol{x}|$, $|\boldsymbol{y}\rangle\langle \boldsymbol{x}|$,

Formalism

n-level system $\leftrightarrow$ n perfectly distinguishable values

Formalism based on $H_n \simeq \mathbb{C}^n$ (n-dimensional Hilbert space)

- Hermitian inner product $\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = x_1^* y_1 + \ldots + x_n^* y_n$
- Norm $||\boldsymbol{x}|| = \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle}$
- Ket-vector $|\boldsymbol{x}\rangle = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$
- Bra-vector $\langle \boldsymbol{x}| = (|\boldsymbol{x}\rangle)^* = (x_1^*, \ldots, x_n^*)$
- Adjoint matrix: $(A^*)_{ij} = A_{ji}^*$ for $m \times n$ matrix A

Formalism

- Trace: Tr$(A) = \sum_{i=1}^{n} A_{ii}$
- For orthonormal basis $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$,
  Tr$(A) = \sum_{i=1}^{n} \langle \boldsymbol{x}_i \mid A\boldsymbol{x}_i \rangle$
- Positivity: $A \geq 0$ iff $(\forall \boldsymbol{x})\ \langle \boldsymbol{x} \mid A\boldsymbol{x} \rangle \geq 0$
- Self-adjointness: $A^* = A$
- Unitarity: $UU^* = U^*U = I$
- Normality: $A^*A = AA^*$

Formalism

Kronecker product:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1s} \\ a_{21} & a_{22} & \dots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rs} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1u} \\ b_{21} & b_{22} & \dots & b_{2u} \\ \vdots & \vdots & \ddots & \vdots \\ b_{t1} & b_{t2} & \dots & b_{tu} \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1s}B \\ a_{21}B & a_{22}B & \dots & a_{2s}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1}B & a_{r2}B & \dots & a_{rs}B \end{pmatrix}$$

Formalism

- $$|\boldsymbol{x}\rangle\langle\boldsymbol{y}| = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \otimes (y_1^*, \ldots, y_n) =$$
$$\begin{pmatrix} x_1 y_1^* & x_1 y_2^* & \ldots & x_1 y_n^* \\ \vdots & \vdots & \ddots & \vdots \\ x_n y_1^* & x_n y_2^* & \ldots & x_n y_n^* \end{pmatrix}$$
- $|\boldsymbol{x}\rangle\langle\boldsymbol{y}|\ |\boldsymbol{z}\rangle = \langle\boldsymbol{y} \mid \boldsymbol{z}\rangle\ |\boldsymbol{x}\rangle$
- If especially $||\boldsymbol{x}|| = 1$, $|\boldsymbol{x}\rangle\langle\boldsymbol{x}|$ is a projection onto a subspace generated by $\boldsymbol{x}$.

Formalism

- Each normal A has spectral representation

$$A = \lambda_1 \, |\boldsymbol{x}_1\rangle\langle\boldsymbol{x}_1| + \ldots + \lambda_n \, |\boldsymbol{x}_n\rangle\langle\boldsymbol{x}_n|,$$

  where $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ is an orthonormal basis of H_n and $\lambda_1, \ldots, \lambda_n$ the eigenvalues of A.
- If A is self-adjoint, each $\lambda_i \in \mathbb{R}$
- If A is unitary, each λ_i has $|\lambda_i| = 1$
- If A is positive, each $\lambda_i \geq 0$.
- Tr$(A) = \lambda_1 + \ldots + \lambda_n$.

Structure of Quantum Mechanics

- State of a physical system: Unit-trace, positive operator T:

$$T = \lambda_1 \, |\boldsymbol{x}_1\rangle\langle\boldsymbol{x}_1| + \ldots + \lambda_n \, |\boldsymbol{x}_n\rangle\langle\boldsymbol{x}_n|,$$

  where $\lambda_i \geq 0$, $\lambda_1 + \ldots + \lambda_n = 1$ (density matrix).

- Observable: Self-adjoint operator A:

$$A = \mu_1 \, |\boldsymbol{y}_1\rangle\langle\boldsymbol{y}_1| + \ldots + \mu_n \, |\boldsymbol{y}_n\rangle\langle\boldsymbol{y}_n|,$$

  where $\mu_i \in \mathbb{R}$ are the potential values of A

- Minimal interpretation:

$$\mathbb{P}(\mu_i) = \text{Tr}(T \, |\boldsymbol{y}_i\rangle\langle\boldsymbol{y}_i|)$$

  is the probability of seeing value μ_i if A is observed when the system is in state T.