The Topology Of Non-Locality and Contextuality

Samson Abramsky
Joint work with Adam Brandenburger

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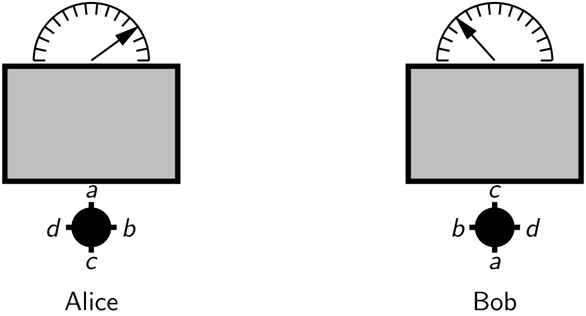
Also new results on local hidden-variable realizations using **negative probabilities**.

The Basic Scenario



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a	b	0	$1/2$	$1/2$	0
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Each row of the table specifies a **probability distribution** on events O^C for a given choice of measurements C.

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Given a family $\{U_i\}_{i \in I}$ with $\bigcup_{i \in I} U_i = U$ and a family of sections $\{s_i \in \mathcal{E}(U_i)\}_{i \in I}$, which is **compatible**: *i.e.* for all $i, j \in I$,

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This says that we can glue together local data which is compatible in the sense of agreeing on overlaps. This is the **sheaf property** — $\mathcal{E}$ is a sheaf.

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We can compose this functor with the event sheaf $\mathcal{E} : \mathcal{P}(X)^{\mathsf{op}} \longrightarrow \mathbf{Set}$, to form a presheaf $\mathcal{D}_R\mathcal{E} : \mathcal{P}(X)^{\mathsf{op}} \longrightarrow \mathbf{Set}$.

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Thus $d|U$ is the **marginal** of the distribution d, which assigns to each section s in the smaller context U the sum of the weights of all sections s' in the larger context which restrict to s.

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This says that the probability for Alice to get the outcome $s_0(m_a)$ is the same, whether we marginalize over the possible outcomes for Bob with measurement m_b, or with m'_b.

In other words, Bob's choice of measurement cannot influence Alice's outcome.

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If d is a global section for the model $\{e_C\}$, we recover the predictions of the model by **averaging over the values of these hidden variables**:

$$e_C(s) \;=\; d|C(s) \;=\; \sum_{s' \in \mathcal{E}(X), s'|C = s} d(s') \;=\; \sum_{s' \in \mathcal{E}(X)} \delta_{s'|C}(s) \cdot d(s').$$

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So:

existence of a local hidden-variable model for a given empirical model
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Hence:

No such h.v. model exists (the empirical model is **non-local**/**contextual**)
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there is an **obstruction to the existence of a global section**

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Bell scenarios (n, k, l): matrix is $(kl)^n \times l^{kn}$.

Existence of Global Sections

Linear algebraic method.

Define system of linear equations $\mathbf{MX} = \mathbf{V}$.

$$\text{Solutions} \quad \longleftrightarrow \quad \text{Global sections}$$

Incidence matrix $\mathbf{M}$ (0/1 entries). Depends only on $\mathcal{M}$ and $\mathcal{E}$.

Enumerate $\coprod_{C \in \mathcal{M}} \mathcal{E}(C)$ as $s_1, \ldots, s_p$.

Enumerate O^X as $t_1, \ldots, t_q$.

$$\mathbf{M}[i,j] = 1 \iff t_j|C = s_i \quad (s_i \in \mathcal{E}(C)).$$

Bell scenarios (n, k, l): matrix is $(kl)^n \times l^{kn}$.

Incidence matrix for $(2, 2, 2)$ is 16×16.

The $(2, 2, 2)$ Incidence Matrix

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$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
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For $(3, 2, 2)$ the matrix is 64×64, rank $= 27$. (Generated by script!)

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Hence solutions correspond exactly to global sections — which as we have seen, correspond exactly to local hidden-variable realizations!

The Bell Model

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	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
(a,b)	0	$1/2$	$1/2$	0
(a',b)	$3/8$	$1/8$	$1/8$	$3/8$
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Solutions in the non-negative reals: this corresponds to solving the linear system over $\mathbb{R}$, subject to the constraint that $\mathbf{X} \geq \mathbf{0}$ (linear programming problem).

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X_9 & + & X_{10} & + & X_{11} & + & X_{12} & = & 1/2 \\
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Adding the last three equations yields

$$X_2 + X_3 + X_4 + X_6 + X_9 + X_{10} + 2X_{11} + X_{12} + X_{13} + X_{14} + X_{15} \; = \; 3/8.$$

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Since all these numbers must be non-negative, the left-hand side of this equation must be greater than or equal to the left-hand side of the first equation, yielding the required contradiction. □

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We consider the possibilistic version of the Hardy model, specified by the following table.

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Now we are interested in solutions over the **boolean semiring**, *i.e.* a boolean satisfiability problem. E.g. the equation specified by the first row of the incidence matrix gives the clause

$$X_1 \ \vee \ X_2 \ \vee \ X_3 \ \vee \ X_4$$

while the fifth yields the formula

$$\neg X_1 \ \wedge \ \neg X_3 \ \wedge \ \neg X_5 \ \wedge \ \neg X_7 .$$

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Since every disjunct in the first formula appears as a negated conjunct in one of the other three formulas, there is no satisfying assignment. □

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Let $\mathbf{V}$ *be the vector over* $\mathbb{R}_{\geq 0}$ *for a probabilistic model,* $\mathbf{V}_b$ *the boolean vector obtained by replacing non-zero elements of* $\mathbf{V}$ *by* 1*. If* $\mathbf{MX} = \mathbf{V}$ *has a solution over* $\mathbb{R}_{\geq 0}$*, then* $\mathbf{MX} = \mathbf{V}_b$ *has a solution over the booleans.*

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Conclusion: Bell $<$ Hardy.

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Note that there is no semiring homomorphism from the *reals* to the booleans or the non-negative reals.

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Proposition

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This vector can be taken as giving a **local hidden-variable realization of the PR box using negative probabilities**. Similar realizations can be given for the other PR boxes.

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Note that, since every no-signalling model of (2,2,2)-type is a convex combination of PR-boxes, it follows from this that all such models, which include all those arising from quantum mechanics, have local hidden variable realizations using negative probabilities.

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Combining this result with Proposition 5, we have the following result.

Proposition

Probabilistic models of type $(2, 2, 2)$ have local hidden-variable realizations with negative probabilities if and only if they satisfy no-signalling.

Strong Contextuality

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Given an empirical model e, we define the set

$$S_e \ := \ \{s \in \mathcal{E}(X) : \forall C \in \mathcal{M}.\, s|C \in \mathsf{supp}(e_C)\}.$$

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We say that the model e is **strongly contextual** if this set S_e is *empty*. Thus strong non-contextuality implies non-extendability.

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$$S_e \ := \ \{s \in \mathcal{E}(X) : \forall C \in \mathcal{M}. \, s|C \in \mathsf{supp}(e_C)\}.$$

A consequence of the extendability of e is that S_e is non-empty.

We say that the model e is **strongly contextual** if this set S_e is *empty*. Thus strong non-contextuality implies non-extendability.

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We shall now show that the well-known GHZ models, of type $(n, 2, 2)$ for all $n > 2$, are strongly contextual. This will establish a strict hierarchy

$$\text{Bell} < \text{Hardy} < \text{GHZ}$$

of increasing strengths of obstructions to non-contextual behaviour for these salient models.

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NB: a model with these properties can be realized in quantum mechanics.

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Thus for any $Y^{(i)}$, $Y^{(j)}$ assigned the **same** value, if we substitute X's in those positions they must receive **different** values. Similarly, for any $Y^{(i)}$, $Y^{(j)}$ assigned different values, the corresponding $X^{(i)}$, $X^{(j)}$ must receive the same value.

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Suppose not all $Y^{(i)}$ are assigned the same value. Then for some i, j, k, $Y^{(i)}$ is assigned the same value as $Y^{(j)}$, and $Y^{(j)}$ is assigned a different value to $Y^{(k)}$. Thus $Y^{(i)}$ is also assigned a different value to $Y^{(k)}$. Then $X^{(i)}$ is assigned the same value as $X^{(k)}$, and $X^{(j)}$ is assigned the same value as $X^{(k)}$. By transitivity, $X^{(i)}$ is assigned the same value as $X^{(j)}$, yielding a contradiction.

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The remaining cases are where all Y's receive the same value. Then any pair of X's must receive different values. But taking any 3 X's, this yields a contradiction, since there are only two values, so some pair must receive the same value.

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This is suggestive of a novel and interesting link between **quantitative** and **qualitative** properties of multipartite non-locality and entanglement.

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- Interplay between abstract mathematics, foundations of physics, and computational exploration.