# Weighted Finite Automata

## Computing with Different Topologies

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Unconventional Computation 2011

- Berstel (et al.)
- Culik II (S. Dube)
- Karhumäki and Kari (et al.)
- M. Latteux (et al.)

- Karhumäki and Kari, A chapter in Handbook of Automata (EMS, to appear)

Unconventional computing

vs. ?

Finite Automata

Answer: to be shown

Unconventional computing

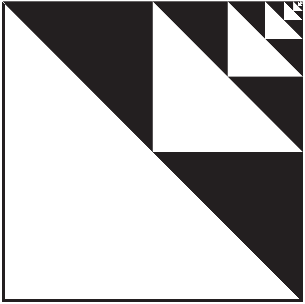
vs. ?

Finite Automata

Answer: to be shown

# Outline

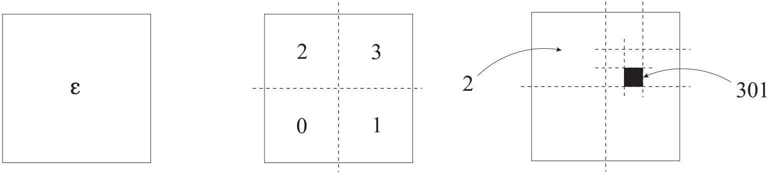
# Computation of Sierpinski's triangle



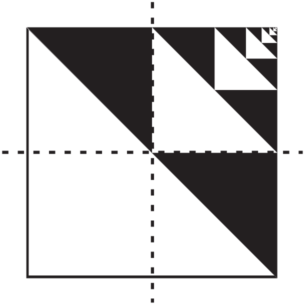
- Encode the picture as a Finite Automaton

# Computation of Sierpinski's triangle

- Addresses: $A = \{0, 1, 2, 3\}$

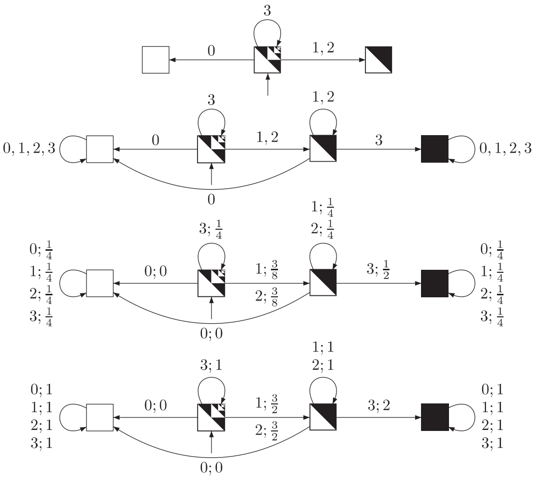


# Computation of Sierpinski's triangle



- Subsquares:

# The resulting automaton

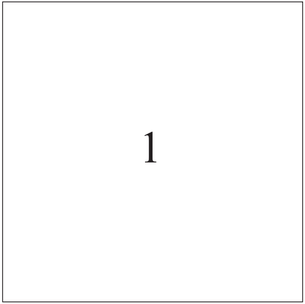



Saturated!

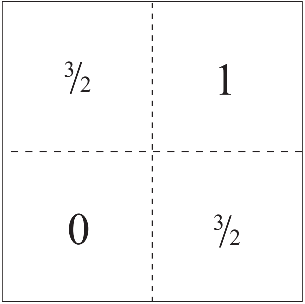
Weights!

Scaling!

# Approximations



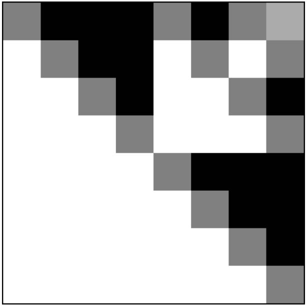
# Approximations



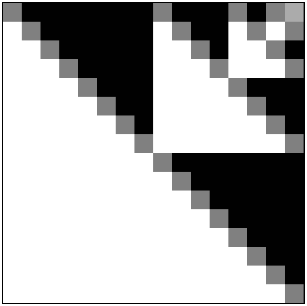
# Approximations

<table>
<tr><td>2/3</td><td>3</td><td>2/3</td><td>1</td></tr>
<tr><td>0</td><td>2/3</td><td>0</td><td>2/3</td></tr>
<tr><td rowspan="2" colspan="2">0</td><td>2/3</td><td>3</td></tr>
<tr><td>0</td><td>2/3</td></tr>
</table>

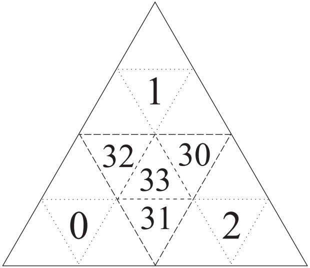
# Approximations

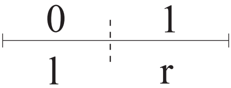


# Approximations



# Other division patterns

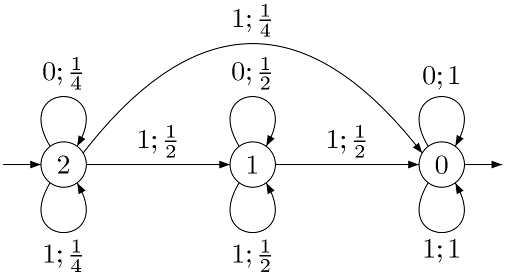


# Weighted Finite Automaton (WFA)

- Nondeterministic finite automaton
- On infinite computations
- Acyclic (level automaton)
- Weights in $\mathbb{R}_+$

# Example




$$M_0 = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_1 = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \quad M_{01} = M_0 M_1 = \begin{pmatrix} \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix}$$

# Weighted Finite Automaton (WFA)

Functions computed: $A = \{0, 1\}$

- $F_{\mathcal{A}} : A^* \to \mathbb{R}_+, \quad IW_w T$
- $f_{\mathcal{A}} : A^\omega \to \mathbb{R}_+, \quad \lim_{n\to\infty} IW_{\mathrm{pref}_n w} T$

The existence of the limit:

- Level automaton
- Weights of the nonterminal loops $< 1$
- Weights of terminal loops $= 1$

# Weighted Finite Automaton (WFA)

## Fact

(i) $F_{\mathcal{A}}$ *and* $f_{\mathcal{A}}$ *are well defined*

(ii) $f_{\mathcal{A}}$ *is continuous or even uniformly continuous (w.r.t.* $d(u,v) = 2^{-|u \wedge v|}$*)*

# Weighted Finite Automaton (WFA)

- Real functions $\hat{f}_{\mathcal{A}} : [0, 1) \to \mathbb{R}_+$
  $A = \{0, 1\}$

$$w = a_1 a_2 a_3 \ldots \in A^\omega$$
$$x = 0, a_1 a_2 a_3 \ldots \in [0, 1)$$

$\therefore\ \text{bin}\, x = w$

- $u10^\omega$ and $u01^\omega$ represent the same number
- Unconventionality!

# Weighted Finite Automaton (WFA)

- $X = A^\omega \setminus A^* 1^\omega = \{w \in A^\omega \mid w \text{ contains infinitely many 0's}\}$
  $w \in X \ : \ \exists \hat{w}$ s.t. $\text{bin}(\hat{w}) = w$
- Dyadic rationals: finite representations

$$\therefore \ \hat{f}_{\mathcal{A}}(x) = f_{\mathcal{A}}(\text{bin } x)$$

# Continuity

When is $\hat{f}_{\mathcal{A}}$

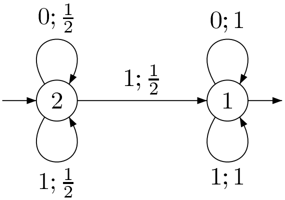
a) continuous

b) smooth?

**Fact**

*$\hat{f}_{\mathcal{A}}$ is continuous in non-dyadic points and right continuous in dyadic points.*

# Example: $\mathcal{A}_1$



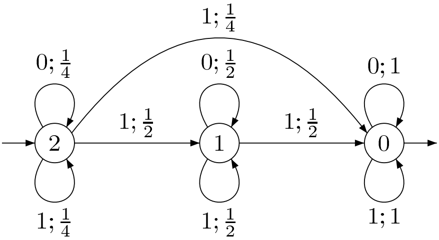

$\mathcal{A}_1$

$$f_{\mathcal{A}_1}(0110^\omega) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$

$$f_{\mathcal{A}_1}(0101^\omega) = \frac{1}{4} + \frac{1}{16}\sum_{i=0}^{\infty}\frac{1}{2^i} = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$

$$\therefore\ \hat{f}_{\mathcal{A}_1}(x) = x$$

# Example: $\mathcal{A}_2$




$\mathcal{A}_2$

$$f_{\mathcal{A}_2}(0w) = \frac{1}{4} f_{\mathcal{A}_2}(w)$$

$$f_{\mathcal{A}_2}(1w) = \frac{1}{4} f_{\mathcal{A}_2}(w) + \frac{1}{2} f_{\mathcal{A}_1}(w) + \frac{1}{4}$$

# Example: $\mathcal{A}_2$

That is

$$\hat{f}_{\mathcal{A}_2}\left(\frac{1}{2}x\right) = \frac{1}{4}\hat{f}_{\mathcal{A}_2}(x)$$

$$\hat{f}_{\mathcal{A}_2}\left(\frac{1}{2}x + \frac{1}{2}\right) = \frac{1}{4}\hat{f}_{\mathcal{A}_2}(x) + \frac{1}{2}\hat{f}_{\mathcal{A}_1}(x) + \frac{1}{4}$$

$$\hat{f}_{\mathcal{A}_1}(1) = 1$$

$$\therefore\ \hat{f}_{\mathcal{A}_2}(x) = x^2$$

# Continuity at $x = \frac{1}{2}$

$$f_{\mathcal{A}}(01^{\omega}) \stackrel{?}{=} f_{\mathcal{A}}(10^{\omega})$$

**Theorem**

*$\hat{f}_{\mathcal{A}}$ is continuous in $[0,1)$ iff $f_{\mathcal{A}}(u01^{\omega}) = f_{\mathcal{A}}(u10^{\omega})$ for all $u \in A^*$.*

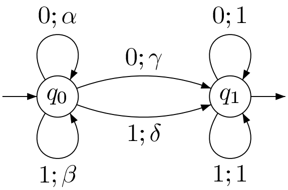
# Continuity vs. initial distribution?

- Strongly continuous: continuous on all initial distributions

**Theorem**

*$\hat{f}_{\mathcal{A}}$ is strongly continuous on the interval $[0,1)$ iff $f_{\mathcal{A}_I}(01^\omega) = f_{\mathcal{A}_I}(10^\omega)$ on all initial distributions $I$ iff $f_{\mathcal{A}_I}(01^\omega) = f_{\mathcal{A}_I}(10^\omega)$ on all coordinate vectors.*

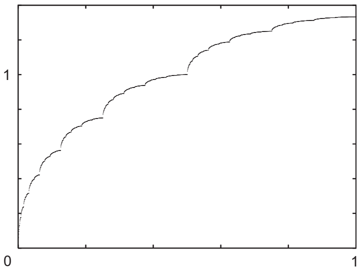
# Example



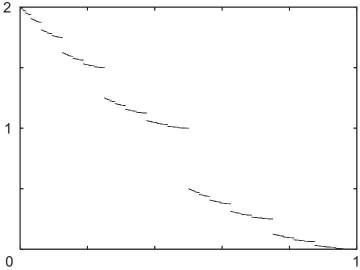

$$f_{\mathcal{A}}(01^\omega) = f_{\mathcal{A}}(10^\omega)$$
$$\Updownarrow$$
$$(\alpha + \beta - 1)\big(\delta(1-\alpha) - \gamma(1-\beta)\big) = 0$$
$$\Updownarrow$$
$$\alpha + \beta = 1 \text{ or } \delta(1-\alpha) = \gamma(1-\beta)$$

# Example

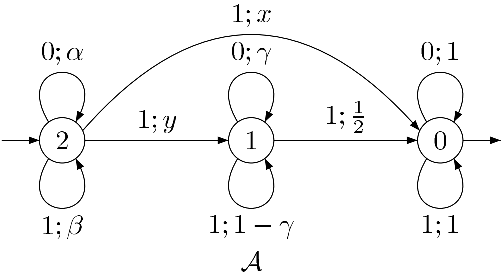
$$\alpha = \frac{3}{4}, \beta = \frac{1}{4}, \gamma = 0, \delta = 1$$



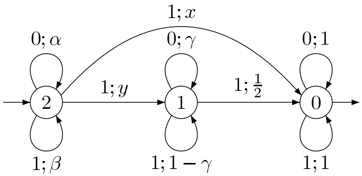

$$\alpha = \frac{1}{2}, \beta = \frac{1}{4}, \gamma = 1, \delta = 0$$



# Computing the parabola



# Computing the parabola




$\mathcal{A}$

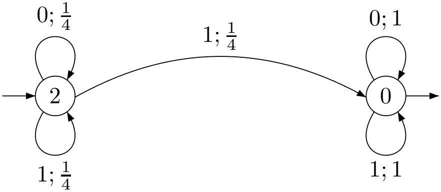
To compute the parabola:

$$\alpha = \beta = \frac{1}{4}$$
$$\gamma = \frac{1}{2}$$

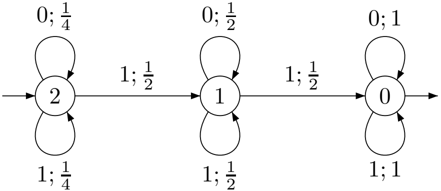
Ratio $\frac{x}{y}$ fixed!

$\therefore$ Unique up to the manipulation of weight on line of length 2!
If minimal!

# Automata-theoretic decomposition

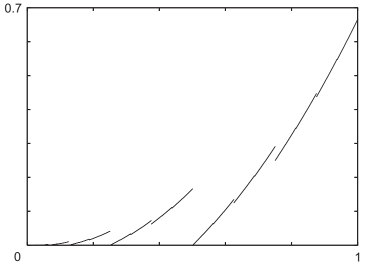


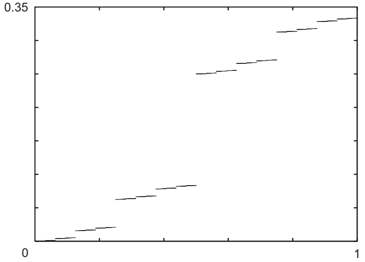

$\mathcal{A}_1'$




$\mathcal{A}_2'$

# Automata-theoretic decomposition

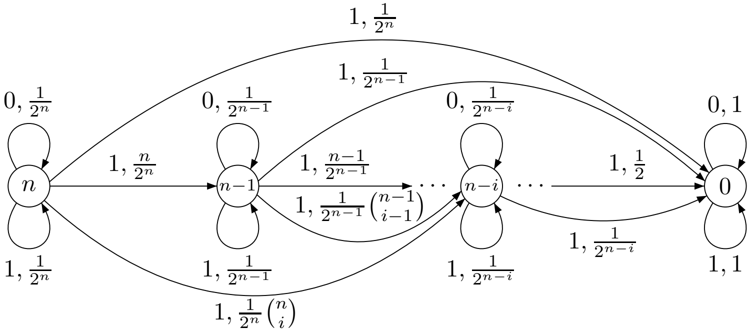



$$\hat{f}_{\mathcal{A}} = \hat{f}_{\mathcal{A}'_2} + \hat{f}_{\mathcal{A}'_1}$$

- Both non-continuous!

# Computing the polynomials




$$\hat{f}_{\mathcal{A}}(x) = x^n$$

# Computing the polynomials

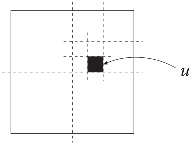
**Theorem**

*Any smooth function computed by a WFA is a polynomial.*

# Image manipulations

Easy:

- Complementation
- Zooming: $f_{\mathcal{A}_{\mathrm{zoom}(u)}}(w) = f_{\mathcal{A}}(uw)$ ($u$ is the address of a pixel)
- Rotations
- Integration: $\hat{f}_{\mathcal{A}_I}(\hat{w}) = \int_0^x \hat{f}_{\mathcal{A}}(\hat{w})$

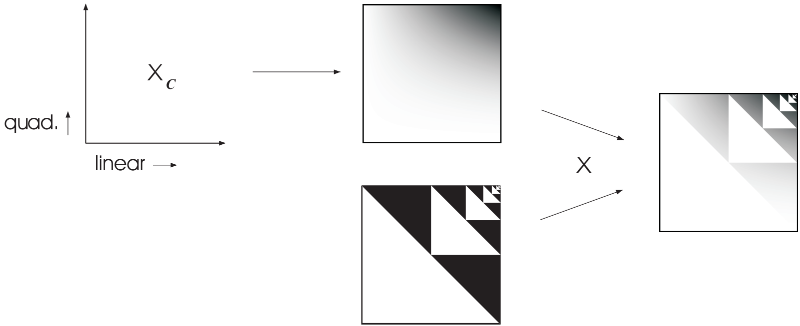


# Products of automata

- Pointwise: (Hadamard product) $f_{\mathcal{A}_1 \cdot \mathcal{A}_2}(w) = f_{\mathcal{A}_1}(w) f_{\mathcal{A}_2}(w)$
- Convolution: (Cauchy product) $f_{\mathcal{A}_1 \times_c \mathcal{A}_2}(w) = \sum_{uv=w} f_{\mathcal{A}_1}(u) f_{\mathcal{A}_2}(v)$
- Complete direct product: $\hat{f}(x,y) = x \cdot y^2$

$$(p, p', i, j, q, q') \xrightarrow{(i,j)} W(p, i, p') \cdot W(q, j, q')$$

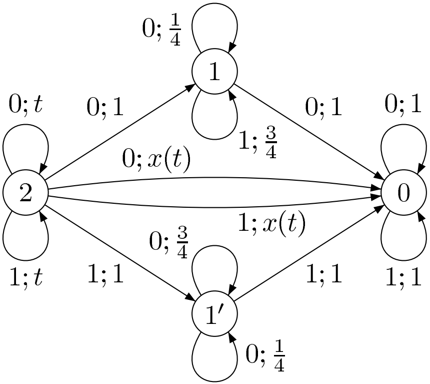
# Products of automata



# Example

$$\hat{f}_{\mathcal{A}}(x, y, z) = (x + 1)^i y^j (z + 2)^k$$

# A monster function

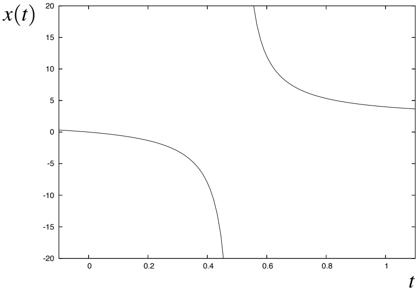


# A monster function

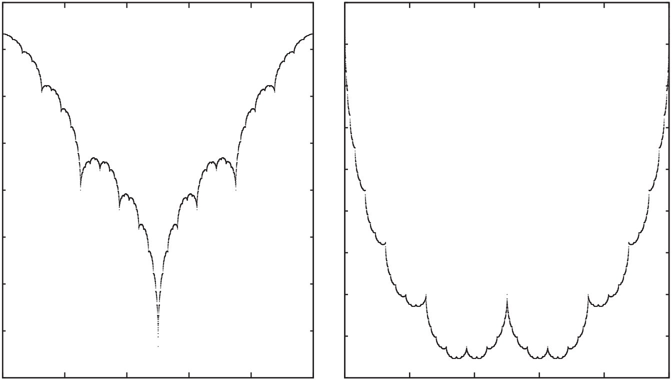
$t$ vs. $x(t)$ when upper half continuous:

$$f_{\mathcal{A}_1}(10^\omega) = f_{\mathcal{A}_1}(01^\omega)$$

$\therefore\ x(t) = \frac{4t}{2t-1}$ $\qquad t \neq \frac{1}{2}$ unique

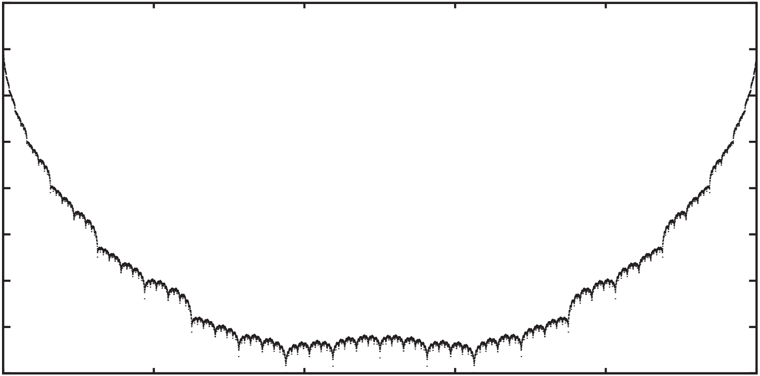


## $t = \frac{1}{4}, t = \frac{3}{4}$



- Continuous, but derivative $= 0$ at dyadic points

# $t = \frac{2}{3}$



- Continuous, but no derivatives! Actually true for all $t \neq \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$

# A monster function

$\mathcal{A}$ contains 4 states $\Rightarrow$

Computational complexity $\sim$ complexity of computing the $n$th power of $4 \times 4$ matrices $\sim$ computational complexity of computing the value of a cubic polynomial!

Conventional activity

..with assistants

Thank you!